Lecture 4.
Scattering from magnetic impurities

Kondo effect on magnetic impurities.
Abrikosov-Suhl resonance.
Spin-orbit interaction.
Spin relaxation.
Electron with spin

Hamiltonian in magnetic field (non-relativistic)

\[ H = -\frac{\hbar^2}{2m} \left( \nabla - \frac{ie}{\hbar c} A \right)^2 + V(r) - \frac{1}{2} g \mu_B \vec{\sigma} \cdot \vec{H} \]

\( \vec{H} \) is Zeeman term

where \( \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \)

\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

Pauli matrices

Non-relativistic also means: no SO interaction

If \( \vec{H} \parallel \hat{z} \)

\[ \vec{\sigma} \cdot \vec{H} = \sigma_z H = \begin{pmatrix} H & 0 \\ 0 & -H \end{pmatrix} \]

Up and down spin are separated

If \( \vec{H} \) is not along \( z \): wave function is an eigenfunction of \( (\vec{\sigma}, \vec{H}) \) operator
Scattering from magnetic impurities

Spin-dependent part

\[ V_s(z) = -\frac{j}{n} \sum_{i} \langle \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_i \rangle \delta(\mathbf{r} - \mathbf{R}_i) \]

\[ n = \frac{N_0}{S^2} \]

\( \hat{\mathbf{S}}_i \) are not uniformly oriented

For one impurity:

If we take \( \hat{\mathbf{S}}_i \) along \( \hat{z} \) axis:

\[ V_{s+}(z) = -\frac{j \hat{S}_z}{n} \delta(\mathbf{r} - \mathbf{R}_i) \]

\[ V_{s-}(z) = +\frac{j \hat{S}_z}{n} \delta(\mathbf{r} - \mathbf{R}_i) \]

\( \Rightarrow \) spin-dependent scattering

Scattering from one impurity

Amplitude of scattering \( \lambda \rightarrow \lambda' \):

\[ t_{\lambda \lambda'} = \frac{j}{n} \langle \hat{\mathbf{S}} \rangle \delta_{\lambda \lambda'} \]

- one electron is scattered from impurity

Probability of scattering to all possible states (Born approximation)

\[ W = \frac{1}{2} \left( \frac{j}{n} \right)^2 \sum_{\lambda \lambda'} \langle \hat{\mathbf{S}}_{\lambda'} \rangle \langle \hat{\mathbf{S}}_{\lambda} \rangle \delta_{\lambda \lambda'} = \frac{1}{2} \left( \frac{j}{n} \right)^2 \sum_{\lambda \lambda'} \langle \hat{\mathbf{S}}_{\lambda'} \rangle \langle \hat{\mathbf{S}}_{\lambda} \rangle \delta_{\lambda \lambda'}. \]
\[ = \frac{1}{2} \left( \frac{\hbar}{n} \right)^2 \sum_{j, k} \left( \delta_{ij} + i \varepsilon_{ijk} \sigma^k \right) \alpha_{\alpha} S_i \cdot S_j = \]
\[ = \left( \frac{\hbar}{n} \right)^2 \bar{s}^2 = \left( \frac{\hbar}{n} \right)^2 s(s+1) \]

**Orders of magnitude:**
- Potential scattering: \( V(\bar{s}) \sim \varepsilon_f \)
- Spin-dependent scattering: \( j < V \) or \( j \ll V \)

**Beyond Born approximation**

**Perturbation theory.**

1) Transitions \( \bar{K} \bar{\Lambda} \rightarrow \bar{K}_1 \bar{\Lambda}_1 \) (intermediate) \( \rightarrow \bar{K}' \bar{\Lambda}' \) (final)

\[ \rightarrow \leftarrow \]

\[ \frac{(a)}{d x, \bar{x}, \alpha} = \left( \frac{\hbar}{n} \right)^2 \sum_{x, \alpha} \frac{d^3 k_1}{(2\pi)^3} \frac{(\vec{\sigma}, \vec{S})_{\alpha x} (\vec{\sigma}, \vec{\bar{S}})_{\bar{x} \bar{\alpha}}}{\varepsilon(K) - \varepsilon(K_1)} (1 - f_{K_1}) \]

where \( \varepsilon(K) = \frac{\hbar^2 k^2}{2m} \)

2) Transition \( \bar{K}_1 \bar{\Lambda}_1 \rightarrow \bar{K}' \bar{\Lambda}' \) and then \( \bar{K} \bar{\Lambda} \rightarrow \bar{K}_1 \bar{\Lambda}_1 \)

\[ \rightarrow \leftarrow \]

\[ \frac{(b)}{d x, \bar{x}, \alpha} = - \left( \frac{\hbar}{n} \right)^2 \sum_{x, \alpha} \frac{d^3 k_1}{(2\pi)^3} \frac{(\vec{\sigma}, \vec{S})_{\alpha x} (\vec{\sigma}, \vec{\bar{S}})_{\bar{x} \bar{\alpha}}}{\varepsilon(K_1) - \varepsilon(K')} f_{K_1} \]

\[ \sum_{x, \alpha} \frac{(\vec{\sigma}, \vec{S})_{\alpha x} (\vec{\sigma}, \vec{\bar{S}})_{\bar{x} \bar{\alpha}}}{\varepsilon(K_1) - \varepsilon(K')} \]

\[ \frac{(c)}{d x, \bar{x}, \alpha} = \frac{(\delta_{ij} + i \varepsilon_{ijk} \sigma^k)_{\alpha \bar{\alpha}}}{d x, \bar{x}, \alpha} S_i \cdot S_j = \]

\[ = \bar{s}^2(s(s+1)) \frac{(\delta_{ij} + i \varepsilon_{ijk} \sigma^k)_{\alpha \bar{\alpha}}}{d x, \bar{x}, \alpha} \]
\[
\sum_{\alpha_1} \left( \vec{\sigma}, \vec{S} \right)_{\alpha_1 \alpha} \left( \vec{\sigma}, \vec{S} \right)_{\alpha_1 \alpha'} = \left( \delta_{\alpha'}^{\alpha} + i \epsilon_{\delta_{\alpha}^{\alpha} \delta_{\alpha'}^{\alpha}} \sigma^k \right)_{\alpha_1 \alpha} \vec{S}_i \cdot \vec{S}_j = \delta_{\alpha_1 \alpha} S(S+1) - \left( \vec{\sigma}, \vec{S} \right)_{\alpha_1 \alpha}
\]

Elastic scattering: \( E(k) = E(k') \)

\[
t^{(s)}_{\alpha \alpha'} = t^{(s)}_{\alpha \alpha} + t^{(s)}_{\alpha \alpha'} = \left( \frac{\hbar}{2m} \right)^2 \sum_{\alpha_1} \int \frac{d^3 k_1}{(2\pi)^3} \frac{\left( \vec{\sigma}, \vec{S} \right)_{\alpha_1 \alpha} \left( \vec{\sigma}, \vec{S} \right)_{\alpha_1 \alpha'} (1 - f_{k_1}) + \left( \vec{\sigma}, \vec{S} \right)_{\alpha_1 \alpha} \left( \vec{\sigma}, \vec{S} \right)_{\alpha_1 \alpha'} f_{k_1}}{E(k) - E(k)}
\]

\[
= \left( \frac{\hbar}{2m} \right)^2 \int \frac{d^3 k_1}{(2\pi)^3} \left[ \frac{\delta_{\alpha \alpha'} S(S+1)}{E(k) - E(k_1)} + \frac{2f_{k_1} - 1}{E(k) - E(k_1)} \left( \vec{\sigma}, \vec{S} \right)_{\alpha \alpha'} \right]
\]

\[
= \frac{1}{2} \left( \frac{\hbar}{2m} \right)^2 \int_{-\mu}^{\mu} \nu(\xi) d\xi_1 \left[ \frac{\delta_{\alpha \alpha'} S(S+1)}{\xi - \xi_1} + \frac{2f(\xi) - 1}{\xi - \xi_1} \left( \vec{\sigma}, \vec{S} \right)_{\alpha \alpha'} \right]
\]

where \( \xi = E(k) - \mu \)

\[
\xi_1 = E(k_1) - \mu
\]

\[
f(\xi) = \frac{1}{e^{\xi/T} + 1} \quad \Rightarrow \quad 2f(\xi) - 1 = -\tanh \frac{\xi}{2T}
\]

\[
I_1 = \int_{-\mu}^{\mu} \frac{\nu(\xi)}{\xi - \xi_1} d\xi \sim \frac{\xi}{\mu}
\]

\[
\nu(\xi = 0) = \nu_F
\]
\[ T_2 = \int_{-\xi}^{\xi} \frac{2f(\xi_1) - 1}{\xi - \xi_1} \nu(\xi_1) d\xi_1 = \]

\[ = \frac{1}{2} \int_{-\xi}^{\xi} \nu d\xi_1 \left[ 2f(\xi_1) - 1 \right] \times \]

\[ \times \left( \frac{1}{\xi - \xi_1} - \frac{1}{\xi + \xi_1} + \frac{1}{\xi - \xi_1} + \frac{1}{\xi + \xi_1} \right) = \]

\[ = \int_{0}^{\xi} \nu d\xi_1 \left( \frac{2\xi_1}{\xi^2 - \xi_1^2} \right) \left[ 2f(\xi_1) - 1 \right] \approx \int_{0}^{\xi} \frac{d\xi_1}{\xi_1} \left( -\frac{\xi_1}{2\xi} \right) \nu(\xi_1) + \]

\[ + \int_{0}^{\xi} \frac{\nu(\xi_1) d\xi_1}{\xi_1} \approx \frac{24\hbar M}{\pi^2} \frac{M}{T} \]

\[ \Rightarrow \xi = 0 \]

\[ t_{\delta}^{-1} = -\frac{1}{\hbar} \left( \sigma, \bar{\sigma} \right) \delta^{-1} \bar{\sigma} \left[ 1 - \frac{2\gamma E}{h} \frac{M}{\left| \xi_1 \right| T} \right] \]

\[ \approx \xi \neq 0 \]

\[ \left| \xi \right| < M \]

Next-to-Born approximation.

(2nd order of perturbation theory)

For conductivity: probability of scattering \[ W_{\delta} = \left| H_{\delta} \right|^2 \]

\[ \Rightarrow \rho = \rho_0 + \rho_1 \left[ 1 - \frac{2\gamma E}{h} \frac{M}{\left| \xi_1 \right| T} \right] \]

\[ \xi < 0 \]

J. Kondo (1964)

Pure Cu, Ag, Cu
"Improved" perturbation theory:

Parquet diagrams, renormalization group approximation

\[
t = -\frac{\delta}{n} \frac{\delta, 3}{1 + \frac{\delta}{n} \frac{h}{\hbar} \frac{\mu}{\{13, 11\}}}
\]

A. Abrikosov (1965)

If \( \delta < 0 \), there is a divergence in scattering amplitude

\[
1 - \frac{1 \delta}{n} \frac{\mu \hbar}{\hbar} = 0
\]

\( \delta = 0 \)

\[
T_k = \mu \exp\left(-\frac{n}{1\delta V_F}\right)
\]

Kondo temperature

For \( T \lesssim T_k \) perturbation theory does not work

Low-temperature limit: P. Nozières (1974)

Other approaches:


Exact solution: P. Wiegmann (1980)
N. Andrei (1980)
Diagrams in Kondo problem

A. Abrikosov (1965)

\[ a \xrightarrow{\delta a(x)} a' \quad \frac{\delta a(x)}{E - E(K) + i\delta \text{sign}(E-\mu)} \quad \text{Green function of electrons} \]

\[ \beta \xrightarrow{\delta \beta'} \beta' \quad \frac{\delta \beta'(x)}{E - E_0 + i\delta \text{sign}(E-\mu)} \quad \text{Green function of localized spin} \]

\[ \beta' \xrightarrow{\delta \beta''} \beta'' \quad -\frac{J}{n} \beta'(x) \beta''(x') \quad \text{vertex} \]

\underline{Vertex corrections (beyond Born approx.)}

\[ t^{(a)} = -i \left( \frac{\hbar}{n} \right)^2 \int \frac{d\omega_1}{2\pi} \frac{d^3k_1}{(2\pi)^3} \left( \mathbf{S} \cdot \mathbf{S} \right)^2 \frac{1}{\left( E - \omega_1 - E(K) + i\delta \text{sign}(E-\omega_1-\mu) \right)^x} \times \frac{1}{\omega_1 + \omega_1 - E_0 + i\delta} \]

\[ t^{(b)} = -i \left( \frac{\hbar}{n} \right)^2 \int \frac{d\omega_1}{2\pi} \frac{d^3k_1}{(2\pi)^3} \sigma^i \sigma^j \mathbf{S}_i \mathbf{S}_j \frac{1}{\left( E + \omega_1 - E(K) + i\delta \text{sign}(E+\omega_1-\mu) \right)^x} \times \frac{1}{\omega_1 + \omega_1 - E_0 + i\delta} \]
\[ t^{(a)} = \left( \frac{\xi}{\hbar} \right)^2 v_F \left[ S(S+1) - \vec{\sigma} \cdot \vec{\tilde{S}} \right] \hbar \frac{\varepsilon_F}{15} \]

\[ t^{(b)} = -\left( \frac{\xi}{\hbar} \right)^2 v_F \left[ S(S+1) + \vec{\sigma} \cdot \vec{\tilde{S}} \right] \hbar \frac{\varepsilon_F}{15} \]

\[ \Rightarrow \text{Renormalized vertex:} \]

\[
t = \frac{\frac{\xi}{\hbar} \vec{\sigma} \cdot \vec{\tilde{S}}}{1 + \frac{3v_F}{\hbar} \ln \frac{\varepsilon_F}{15}}
\]

Self-energy correction:

\[ \Sigma(3) = -i \text{sign} \xi N_i v_F |t(3)|^2 \]

Abrikosov-Suhl resonance
Spin-orbit interaction

Relativistic term in the Hamiltonian:

$$H_{so} = \frac{\hbar^2}{2m^* c^2} \delta \cdot [\hat{\mathbf{r}} \times \nabla V]$$

Potential $V(r)$ from:
- crystal lattice
- impurities
- interfaces

Example: Electrons near the interface

$$H_{so} = \frac{\hbar^2}{2m^* c^2} \frac{dV}{dz} (\sigma \times \hat{\mathbf{r}}) \cdot$$

$$\rightarrow \lambda_{so} (\sigma_y K_x - \sigma_x K_y)$$

Rashba SO interaction

Spin relaxation: $\tau_{sp}$ - scattering with spin-flip.
Conductivity

\[ \sigma_o = \frac{2e^2}{3\omega} \int \frac{d^3k}{(2\pi)^3} \frac{d\varepsilon}{2\pi} \varepsilon^2 G(K, \varepsilon+i\omega) G(K, \varepsilon) \]

Limit of \( \omega \to 0 \)

\[ \int \frac{d\varepsilon}{2\pi} \frac{1}{\varepsilon+i\omega-\varepsilon_k+i\frac{i}{2\pi}} \frac{1}{\varepsilon-\varepsilon_k+i\frac{i}{2\pi}} \to \]

\[ \int_{\mu-i\omega}^{\mu+i\omega} \frac{d\varepsilon}{2\pi} \frac{1}{\varepsilon+i\omega-\varepsilon_k+i\frac{i}{2\pi}} \frac{1}{\varepsilon-\varepsilon_k+i\frac{i}{2\pi}} = \]

\[ = \frac{\omega}{2\pi} G^R(K, \mu) G^A(K, \mu) \]

\[ \sigma_o = \frac{e^2}{3\pi} \int \frac{d^3k}{2\pi} \frac{d\varepsilon}{2\pi} \varepsilon^2 \frac{1}{\mu-\varepsilon_k+i\frac{i}{2\pi}} \frac{1}{\mu-\varepsilon_k-i\frac{i}{2\pi}} = \]

\[ = \frac{e^2}{3\pi m} \int \nu(\varepsilon_k) d\varepsilon_k \frac{\varepsilon_k}{(\varepsilon_k-\mu-i\frac{i}{2\pi})(\varepsilon_k-\mu+i\frac{i}{2\pi})} = \]

\[ = \frac{e^2}{3\pi m} \nu_F \frac{G}{K} \frac{2\pi k^2}{2\pi} = \frac{2e^2 \nu_F \pi \varepsilon_F}{3m} = \]

\[ \sigma_o = \frac{ne^2}{m} \]

\[ \frac{m^* \beta^2}{2} = \varepsilon_k \]