„Strong interactions with quarks and mesons – on unitarisation including bound states an non-hermitian quantum theory"

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by

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Coupled channel model of confined and scattering system ②

(Nijmegen Unitarised Meson Model)

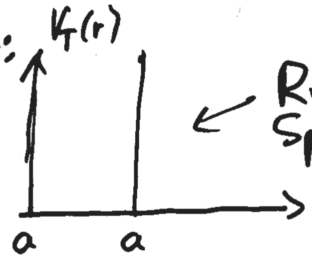
S = „scattering" = Meson-Meson-Scattering

B = „bound" = confined Quark-Antiquark System

T = „transition" = Meson-Quark/Antiquark-transition

$$\boxed{\begin{gathered} K := k_S = \sqrt{2\mu_S\left(E-E_S^{(0)}\right)} \quad , \; k_B = \sqrt{2\mu_B\left(E-E_B^{(0)}\right)} \\ \left(\frac{d^2}{dr^2} - \frac{l_S(l_S+1)}{r^2} - 2\mu_S V_S(r) + k_S^2\right)\psi_S(r) = 2\mu_S V_T(r)\,\psi_B(r) \\ \left(\frac{d^2}{dr^2} - \frac{l_B(l_B+1)}{r^2} - 2\mu_B V_B(r) + k_B^2\right)\psi_B(r) = 2\mu_B V_T^*(r)\,\psi_S(r) \\ V_B(r) = \frac{1}{2}\mu_B\omega^2 r^2 \quad , \; V_S(r) \approx 0 \, , \; \psi_S(0) = \psi_B(0) = 0 \end{gathered}}$$

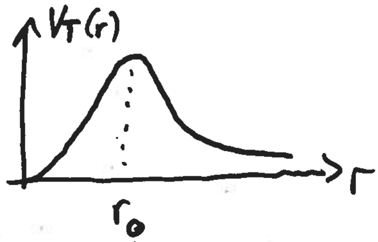
Two versions of $V_T(r)$ in use:



Resonance Spectrum expansion

① $V_T(r) \sim \sqrt{E}\, g\, \delta(r-a)$

② $V_T(r) \sim \sqrt{E}\, g\, \frac{r}{r_0}\, e^{-\frac{1}{2}\left(\frac{r}{r_0}\right)^2}$



≈ 3 contributions of F. Kleefeld (see e.g. hep-ph/0310320)

① Interpretation of dyn. generated poles as resonances of $V_T(r)$

② Mass scaling: $\sqrt{a^2\mu_B}$ universal (e.g. $a_{n\bar{n}}\sqrt{\mu_{n\bar{n}}} = a_{c\bar{c}}\sqrt{\mu_{c\bar{c}}}$)

③ Interpretation of $\sqrt{E}$-Faktor of g → yielding Adler-zero

Motivation & history

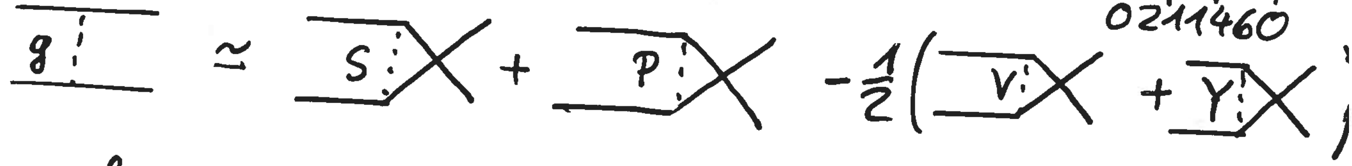
- **Contact with M.D. Scadron**

  $\Rightarrow$ Quark-level-linear-sigma-model

$$\mathcal{L}_{\sigma\bar{q}q} = g\,\bar{\psi}\,\sigma\,\psi \qquad \mathcal{L}_{\sigma^4} = \frac{\lambda}{4}\sigma^4 \qquad \text{NJL: } \underbrace{g^2}_{\sim\lambda}(\bar{\psi}\psi)^2$$

Dynamical generation $\lambda \sim g^2 > 0$

- **Fierzing of QCD (1 gluon exchange)** e.g. hep-ph/0211460



$\Rightarrow$ Linear-σ-model with imaginary coupling ig

$$\mathcal{L}_{\sigma\bar{q}q} = ig\,\bar{\psi}\,\sigma\,\psi \qquad \mathcal{L}_{\sigma^4} = \frac{\lambda}{4}\sigma^4 \qquad \text{NJL: } \underbrace{(ig)^2}_{\sim\lambda}(\bar{\psi}\psi)^2$$

$\left(g \sim \sqrt{\frac{N_F}{N_c}}\right)$ $\qquad \lambda \sim (ig)^2 < 0$

Surprise: Quark-Meson-Lagrangian non-Hermitian!

D. J. Gross & F. Wilczek (PRL 30 (1973) 1343):

"... K Symanzik ... suggested ... a $\lambda\varphi^4$ theory with a negative λ ... . However, one can show, using the renormalization group equations, that in such theory the ground-state energy is unbounded from below (S Coleman, private communication) ..."

H D Politzer (PRL 30 (1973) 1346):

"... $\lambda\varphi^4$ theory with $\lambda < 0$ is ... infrared unstable ... . In particular, the potential whose minimum determines the vacuum decreases without bound for large field ...."

Lars Brink (presentation speech for the Nobel Prize in Physics 2004):

"The theory of Gross, Politzer and Wilczek successfully describes the physics of quarks ... further research has shown, that these theories are unique. ... It is wonderful to know that nature has chosen the only theory that we have found to be possible ..."

See also: F. Kleefeld, J. Phys. A: Math. Gen. 38 (2005) L1-L7 (hep-th/0506142)

- First contact with C. Bender in Kiev

Claim: several Hamiltonians being non-hermitian have real or complex-conjugate spectrum due to PT-symmetry
physics/9712001 $\qquad x \to -x,\ i \to -i$

$$\left.\begin{array}{lll} V(x) \text{ e.g.} & ix^3, & -x^4 \\ & \updownarrow & \updownarrow \\ & i\phi^3 & -\phi^4 \\ & i\bar{q}\phi q & \end{array}\right\}$$

Spectrum bound from below, exp. decaying wave-funktions

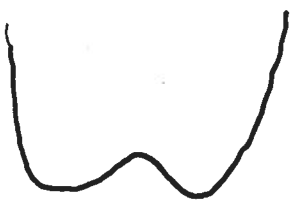
- In the meantime we know:

There exists for PT-symmetric Hamiltonians some non-local metric which makes positive scalar product

$$\langle \psi | \eta \phi \rangle = \int dx\,dy\ \psi^*(x)\,\eta(x,y)\,\phi(y)$$

There holds $\eta^{-1} H \eta = H^+$

By taking the square root of the metric one obtains by equivalence transform a hermitian nonlocal Hamiltonian $h = \sqrt{\eta}^{-1} H \sqrt{\eta}$ with standard scalar product.



Result $-\phi^4$-theory $\xrightarrow[\text{transform}]{\text{equivalence}}$
(Jones quant-ph/0601188) (anomaly)

- <u>Problem:</u> How to construct non-Hermitian Dirac-Lagrangian?

$$\cancel{(ig\,\overline{\psi}\,\sigma\,\psi)^{\dagger} = -ig\,\overline{\psi^*}\,\sigma^*\,\psi^*}$$

<u>Idea:</u> Action must be under transposition a scalar

$$(\psi^T\psi)^T \underset{\substack{\uparrow \\ \text{Grassmanns}}}{=} -\psi^T\psi \quad ↯$$

Better: $(\psi^T C\psi)^T = \underbrace{\psi^T C}_{\overline{\psi^c}}\psi$ with $C^T = -C$

$$\Rightarrow \quad \mathcal{L} = \overline{\psi^c}(i\partial\!\!\!/ + g\sigma - m)\psi = \mathcal{L}^T$$

Klein-Gordon-equation decomposes into two Dirac-equations

$$(-i\partial\!\!\!/ - g\sigma - m)(i\partial\!\!\!/ - g\sigma - m)\psi = 0$$

$$\Rightarrow \quad \psi = \psi^{(+)} + \psi^{(-)} \quad \text{with}$$

CPT

$$(i\partial\!\!\!/ - g\sigma - m)\psi^{(+)} = 0$$

$$(-i\partial\!\!\!/ - g\sigma - m)\psi^{(-)} = 0$$

& Transposed equations

$$\overline{\psi^{(-)c}}(i\overleftarrow{\partial\!\!\!/} - g\sigma - m) = 0$$

$$\overline{\psi^{(+)c}}(-i\overleftarrow{\partial\!\!\!/} - g\sigma - m) = 0$$

Conserved currents

$\Rightarrow$ $0 = \partial_\mu[\overline{\psi^{(+)c}}\gamma^\mu\psi^{(+)}$

even for g, m complex

- **Observation:**

$ig\,\overline{\psi^c}\,\sigma\,\psi$ is of ix^3 type and therefore stable.

Moreover: β-Function negativ
$\Rightarrow$ theory is asymtotic free & non-local
$\Rightarrow$ Quarks can't go on mass-shell and be observed

Idea of quark mass generation:

$\overline{q^c}\, i g_2 \sigma\, q + g_1 \overline{q^c}\, H\, q$ Problem σ-H-mixing

$\sigma \to \sigma + \langle\sigma\rangle$

$H \to H + \langle H\rangle$

$$\Rightarrow \quad m_q = \underbrace{g_1\langle H\rangle}\ + \underbrace{i g_2\langle\sigma\rangle}$$

$\approx 5\,\mathrm{MeV}/c^2 - i\,330\,\mathrm{MeV}/c^2$

No gluons needed!

Questions: Does there exist non-hermitian SUSY?
How about gravitation with complex mass?

Setup of Non-hermitian Quantum Mechanics

Recall: Klein-Gordon equation (wave equation) is differential equation 2nd order in time decomposing in two 1st. order equations:

$$\left((i\hbar\partial_t)^2 - H^2\right)|\psi(t)\rangle = 0 \quad \leftarrow \text{Klein-Gordon-like}$$

$$\Rightarrow (i\hbar\partial_t - H)(i\hbar\partial_t + H)|\psi(t)\rangle = 0$$

$$\Rightarrow \underbrace{(i\hbar\partial_t - H)|\psi^{(+)}(t)\rangle = 0}_{\text{"retarded Schrödinger equ."}} \;,\; \underbrace{(i\hbar\partial_t + H)|\psi^{(-)}(t)\rangle = 0}_{\text{"advanced Schrödinger equ."}}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$i\hbar\partial_t|\psi^{(+)}(t)\rangle = H|\psi^{(+)}(t)\rangle \;,\; -i\hbar\partial_t|\psi^{(-)}(t)\rangle = H|\psi^{(-)}(t)\rangle$$

Respective equations for left eigen-solutions:

$$\underbrace{i\hbar\partial_t \langle\!\langle\psi^{(+)}(t)| = \langle\!\langle\psi^{(+)}(t)|H}_{\text{"transposed retarded Schr. equ."}} \;,\; \underbrace{-i\hbar\partial_t \langle\!\langle\psi^{(-)}(t)| = \langle\!\langle\psi^{(-)}(t)|H}_{\text{"transposed advanced Schr. equ."}}$$

Hermitian conjugation $\langle\!\langle\ldots| = |\ldots\rangle\!\rangle^\dagger$ and $|\ldots\rangle = \langle\ldots|^\dagger$ yields

$$-i\hbar\partial_t \langle\psi^{(+)}(t)| = \langle\psi^{(+)}(t)|H^\dagger \;,\; i\hbar\partial_t\langle\psi^{(-)}(t)| = \langle\psi^{(-)}(t)|H^\dagger$$

$$-i\hbar\partial_t|\psi^{(+)}(t)\rangle\!\rangle = H^\dagger|\psi^{(+)}(t)\rangle\!\rangle \;,\; i\hbar\partial_t|\psi^{(-)}(t)\rangle\!\rangle = H^\dagger|\psi^{(-)}(t)\rangle\!\rangle$$

Transposition and hermitian conjugations yield different results, e.g.:

$$i\hbar\partial_t \langle\!\langle\psi^{(-)}(t)|\psi^{(+)}(t)\rangle = \left(i\hbar\partial_t\langle\!\langle\psi^{(-)}(t)|\right)|\psi^{(+)}(t)\rangle + \langle\!\langle\psi^{(-)}(t)|\left(i\hbar\partial_t|\psi^{(+)}(t)\rangle\right)$$

$$= \langle\!\langle\psi^{(-)}(t)|(-H+H)|\psi^{(+)}(t)\rangle = 0$$

$$i\hbar\partial_t\langle\psi^{(+)}(t)|\psi^{(+)}(t)\rangle = \langle\psi^{(+)}(t)|(-H^\dagger+H)|\psi^{(+)}(t)\rangle \neq 0$$

Non-hermitian Quantum Mechanics → Spatial Representation

$$\vec{\psi}^{(+)}(z,t) = \langle\!\langle z | \vec{\psi}^{(+)}(t) \rangle\!\rangle$$

$$\vec{\psi}^{(-)}(z,t)^T = \langle\!\langle \vec{\psi}^{(-)}(t)^T | z \rangle\!\rangle \qquad T = \text{transposition}$$

⇒ (Stationary) Retarded Schrödinger equation:

$$\boxed{+i\hbar\,\partial_t\, \vec{\psi}^{(+)}(z,t) = \left[-\frac{\hbar^2}{2M}\frac{d^2}{dz^2} + V(z)\right] \vec{\psi}^{(+)}(z,t)}$$

Transposed advanced Schrödinger equation:

$$\boxed{-i\hbar\,\partial_t\, \vec{\psi}^{(-)}(z,t) = \left[-\frac{\hbar^2}{2M}\frac{d^2}{dz^2} + V(z)^T\right] \vec{\psi}^{(-)}(z,t)}$$

Derivation of continuity equation: (→ complex energy flux)

$$i\hbar\, \vec{\psi}^{(-)T}\left(\partial_t \vec{\psi}^{(+)}\right) = -\vec{\psi}^{(-)T}\frac{\hbar^2}{2M}\frac{d^2}{dz^2}\vec{\psi}^{(+)} + \vec{\psi}^{(-)T} V(z)\vec{\psi}^{(+)} \quad (\mathrm{I})$$

$$-i\hbar\left(\partial_t \vec{\psi}^{(-)T}\right)\vec{\psi}^{(+)} = -\left(\frac{d^2}{dz^2}\vec{\psi}^{(-)T}\right)\frac{\hbar^2}{2M}\vec{\psi}^{(+)} + \vec{\psi}^{(-)T} V(z)\psi^{(+)} \quad (\mathrm{II})$$

$$\overset{(\mathrm{I})-(\mathrm{II})}{\Longrightarrow}\; i\hbar\,\partial_t\left(\vec{\psi}^{(-)T}\cdot\vec{\psi}^{(+)}\right) = -\left[\vec{\psi}^{(-)T}\frac{\hbar^2}{2M}\frac{d^2}{dz^2}\vec{\psi}^{(+)} - \left(\frac{d^2}{dz^2}\vec{\psi}^{(-)T}\right)\frac{\hbar^2}{2M}\vec{\psi}^{(+)}\right]$$

$$\Rightarrow \boxed{\partial_t \underbrace{\left(\vec{\psi}^{(-)T}\cdot\vec{\psi}^{(+)}\right)}_{\rho(z,t)} = -\frac{d}{dz}\underbrace{\left[\frac{1}{i\hbar}\left(\vec{\psi}^{(-)T}\frac{\hbar^2}{2M}\frac{d}{dz}\vec{\psi}^{(+)} - \left(\frac{d}{dz}\vec{\psi}^{(-)T}\right)\frac{\hbar^2}{2M}\vec{\psi}^{(+)}\right)\right]}_{j(z,t)}}$$

$$\Rightarrow \partial_t \int_{z_1}^{z_2} dz\, \vec{\psi}^{(-)}(z,t)^T\cdot\vec{\psi}^{(+)}(z,t) = -\left(j(z_2,t) - j(z_1,t)\right)$$

$$\Rightarrow \boxed{\int_{z_1}^{z_2} dz\, \vec{\psi}^{(-)}(z,t)^T\cdot\vec{\psi}^{(+)}(z,t) = \text{konst} \quad \text{for } j(z_2,t) = j(z_1,t)}$$

Recall:

$$i\hbar\partial_t \vec{\psi}^{(+)}(z,t) = \left[-\frac{\hbar^2}{2M}\frac{d^2}{dz^2} + V(z)\right]\vec{\psi}^{(+)}(z,t)$$

$$-i\hbar\partial_t \vec{\psi}^{(-)}(z,t) = \left[-\frac{\hbar^2}{2M}\frac{d^2}{dz^2} + V(z)^T\right]\vec{\psi}^{(-)}(z,t)$$

Separation ansatz: $\vec{\psi}^{(\pm)}(z,t) = e^{\pm\frac{1}{i\hbar}Et}\,\vec{\Phi}_E^{(\pm)}(z)$

$$\Rightarrow \quad E\,\vec{\Phi}_E^{(+)}(z) = \left[-\frac{\hbar^2}{2M}\frac{d^2}{dz^2} + V(z)\right]\vec{\Phi}_E^{(+)}(z)$$

$$E\,\vec{\Phi}_E^{(-)}(z) = \left[-\frac{\hbar^2}{2M}\frac{d^2}{dz^2} + V(z)^T\right]\vec{\Phi}_E^{(-)}(z)$$

Construct complete set of eigenfunctions

$$\left[-\frac{\hbar^2}{2M}\frac{d^2}{dz^2} + V(z)\right]\vec{\Phi}_n^{(+)}(z) = E_n\,\vec{\Phi}_n^{(+)}(z)$$

$$\left[-\frac{\hbar^2}{2M}\frac{d^2}{dz^2} + V(z)^T\right]\vec{\Phi}_m^{(-)}(z) = E_m\,\vec{\Phi}_m^{(-)}(z)$$

$\Rightarrow$ As for continuity equation:

$$-\frac{d}{dz}\Bigg[\underbrace{\frac{1}{i\hbar}\left(\vec{\Phi}_m^{(-)T}\frac{\hbar^2}{2M}\frac{d}{dz}\vec{\Phi}_n^{(+)} - \left(\frac{d}{dz}\vec{\Phi}_m^{(-)T}\right)\frac{\hbar^2}{2M}\vec{\Phi}_n^{(+)}\right)}_{W_{mn}(z)\ \leftarrow\ \text{„Wronskian"}}\Bigg] = \frac{1}{i\hbar}(E_n - E_m)\,\vec{\Phi}_m^{(-)T}\cdot\vec{\Phi}_n^{(+)}$$

$$\Rightarrow -\int_{z_1}^{z_2} dz\,\frac{d}{dz}W_{mn}(z) = -\left[W_{mn}(z_2) - W_{mn}(z_1)\right] = \frac{1}{i\hbar}(E_n - E_m)\int_{z_1}^{z_2} dz\,\vec{\Phi}_m^{(-)T}(z)\cdot\vec{\Phi}_n^{(+)}(z)$$

For $W_{mn}(z_2) = W_{mn}(z_1) \Rightarrow \displaystyle\int_{z_1}^{z_2} dz\,\vec{\Phi}_m^{(-)T}(z)\cdot\vec{\Phi}_n^{(+)}(z) \propto \delta_{mn}$

$\uparrow$ $=$ (Normalisation)

$$\vec{\psi}^{(\pm)}(z,t) = \sum_n c_n^{(\pm)}\,e^{\pm\frac{1}{i\hbar}E_n t}\,\vec{\Phi}_n^{(\pm)}(z)$$

$$\Rightarrow \int_{z_1}^{z_2} dz\,\vec{\psi}^{(-)T}(z,t)\cdot\vec{\psi}^{(+)}(z,t) = \sum_n c_n^{(-)}c_n^{(+)}\overbrace{\int_{z_1}^{z_2} dz\,\vec{\Phi}_n^{(-)T}(z)\,\vec{\Phi}_n^{(+)}(z)}^{1}$$

Free theory

$$\underbrace{\frac{2M}{\hbar^2}E}_{k_0^2}\vec{\Phi}^{(+)}(z) = -\frac{d^2}{dz^2}\vec{\Phi}^{(+)}(z)\,, \quad \underbrace{\frac{2M}{\hbar^2}E}_{k_0^2}\vec{\Phi}^{(-)}(z) = -\frac{d^2}{dz^2}\vec{\Phi}^{(-)}(z)$$

$$\Rightarrow \vec{\Phi}^{(+)}(z) = e^{ik_0z}\vec{c}^{(+)}(k_0) + e^{-ik_0z}\vec{c}^{(+)}(-k_0)$$

$$\vec{\Phi}^{(-)}(z) = e^{-ik_0z}\vec{c}^{(-)}(k_0) + e^{ik_0z}\vec{c}^{(-)}(-k_0)$$

Conserved current:

$$j(z) = \frac{1}{i\hbar}\left[\vec{\Phi}^{(-)T}(z)\frac{\hbar^2}{2M}\cdot\vec{\Phi}^{(+)}(z)' - \vec{\Phi}^{(-)T}(z)'\frac{\hbar^2}{2M}\vec{\Phi}^{(+)}(z)\right]$$

$$= \frac{1}{i\hbar}\left(\vec{\Phi}^{(-)T}(z)\sqrt{\tfrac{\hbar^2}{2M}}, \vec{\Phi}^{(-)T}(z)'\sqrt{\tfrac{\hbar^2}{2M}}\right)\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix}\sqrt{\tfrac{\hbar^2}{2M}}\vec{\Phi}^{(+)}(z) \\ \sqrt{\tfrac{\hbar^2}{2M}}\vec{\Phi}^{(+)}(z)'\end{pmatrix}$$

$$= \vec{c}^{(-)}(k_0)^T\frac{\hbar k_0}{M}\vec{c}^{(+)}(k_0) - \vec{c}^{(-)}(-k_0)^T\frac{\hbar k_0}{M}\vec{c}^{(+)}(-k_0)$$

There holds:

$$\begin{pmatrix}\sqrt{\tfrac{\hbar^2}{2M}}\vec{\Phi}^{(\pm)}(z_>) \\ \sqrt{\tfrac{\hbar^2}{2M}}\vec{\Phi}^{(\pm)}(z_>)'\end{pmatrix} = \underbrace{\begin{pmatrix}\cos(k_0(z_>-z_<)) & \frac{\sin(k_0(z_>-z_<))}{k_0} \\ -k_0\sin(k_0(z_>-z_<)) & \cos(k_0(z_>-z_<))\end{pmatrix}}_{T^{(\pm)} = \begin{pmatrix} T_{11}^{(\pm)} & T_{12}^{(\pm)} \\ T_{21}^{(\pm)} & T_{22}^{(\pm)}\end{pmatrix} = \text{„transfer matrix" (here: free)}}\begin{pmatrix}\sqrt{\tfrac{\hbar^2}{2M}}\vec{\Phi}^{(\pm)}(z_<) \\ \sqrt{\tfrac{\hbar^2}{2M}}\vec{\Phi}^{(\pm)}(z_<)'\end{pmatrix}$$

Current conservation $\Rightarrow j(z_>) = j(z_<)$

$$\Rightarrow T^{(-)T}\begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}T^{(+)} = \begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}$$

Non-hermitian scattering theory

$V(z)=0$ for $z \notin [z_<, z_>]$

Starting point: current conservation $j(z_>) = j(z_<)$

$$\Rightarrow \quad \vec{C}_>^{(-)}(k_0)^T \frac{\hbar k_0}{M} \vec{C}_>^{(+)}(k_0) - \vec{C}_>^{(-)}(-k_0)^T \frac{\hbar k_0}{M} \vec{C}_>^{(+)}(-k_0)$$

$$= \vec{C}_<^{(-)}(k_0)^T \frac{\hbar k_0}{M} \vec{C}_<^{(+)}(k_0) - \vec{C}_<^{(-)}(-k_0)^T \frac{\hbar k_0}{M} \vec{C}_<^{(+)}(-k_0)$$

or equivalently
$$\left.\begin{aligned} &\vec{a}_1^{(-)T} \cdot \vec{a}_1^{(+)} - \vec{e}_2^{(-)T} \cdot \vec{e}_2^{(+)} \\ &= \vec{e}_1^{(-)T} \cdot \vec{e}_1^{(+)} - \vec{a}_2^{(-)T} \cdot \vec{a}_2^{(+)} \end{aligned}\right\}$$

with
$$\vec{e}_1^{(\pm)} = e^{\pm i k_0 z_<} \sqrt{\frac{\hbar k_0}{M}}\, \vec{C}_<^{(\pm)}(k_0)$$
$$\vec{e}_2^{(\pm)} = e^{\mp i k_0 z_>} \sqrt{\frac{\hbar k_0}{M}}\, \vec{C}_>^{(\pm)}(-k_0)$$
$$\vec{a}_1^{(\pm)} = e^{\pm i k_0 z_>} \sqrt{\frac{\hbar k_0}{M}}\, \vec{C}_>^{(\pm)}(k_0)$$
$$\vec{a}_2^{(\pm)} = e^{\mp i k_0 z_<} \sqrt{\frac{\hbar k_0}{M}}\, \vec{C}_<^{(\pm)}(-k_0)$$

Definition of S-matrix:

$$\begin{pmatrix} \vec{a}_1^{(\pm)} \\ \vec{a}_2^{(\pm)} \end{pmatrix} = \underbrace{\begin{pmatrix} S_{11}^{(\pm)} & S_{12}^{(\pm)} \\ S_{21}^{(\pm)} & S_{22}^{(\pm)} \end{pmatrix}}_{S^{(\pm)}} \begin{pmatrix} \vec{e}_1^{(\pm)} \\ \vec{e}_2^{(\pm)} \end{pmatrix} \quad \text{with } S^{(-)T} S^{(+)} = \mathbb{1}$$

$$\begin{pmatrix} S_{11}^{(-)T} & S_{21}^{(-)T} \\ S_{12}^{(-)T} & S_{22}^{(-)T} \end{pmatrix} \begin{pmatrix} S_{11}^{(+)} & S_{12}^{(+)} \\ S_{21}^{(+)} & S_{22}^{(+)} \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \Rightarrow$$

$$\overbrace{S_{11}^{(-)T} S_{11}^{(+)}}^{T_1} + \overbrace{S_{21}^{(-)T} S_{21}^{(+)}}^{R_1} = \mathbb{1}$$

$$\underbrace{S_{12}^{(-)T} S_{12}^{(+)}}_{R_2} + \underbrace{S_{22}^{(-)T} S_{22}^{(+)}}_{T_2} = \mathbb{1}$$

$$S_{11}^{(-)T} S_{12}^{(+)} + S_{21}^{(-)T} S_{22}^{(+)} = 0$$

$$S_{12}^{(-)T} S_{11}^{(+)} + S_{22}^{(-)T} S_{21}^{(+)} = 0$$

1. Step: Determine matrix $\tilde{T}^{(\pm)}$ given by

$$\underbrace{\begin{pmatrix} \vec{a}_1^{(\pm)} \\ \vec{e}_2^{(\pm)} \end{pmatrix}}_{\text{out-state}} = \underbrace{\begin{pmatrix} \tilde{T}_{11}^{(\pm)} & \tilde{T}_{12}^{(\pm)} \\ \tilde{T}_{21}^{(\pm)} & \tilde{T}_{22}^{(\pm)} \end{pmatrix}}_{\tilde{T}^{(\pm)}} \underbrace{\begin{pmatrix} \vec{e}_1^{(\pm)} \\ \vec{a}_2^{(\pm)} \end{pmatrix}}_{\text{in-state}}$$

Recall: Current conservation implies

$$(\vec{a}_1^{(-)T}, \vec{e}_2^{(-)T}) \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} \vec{a}_1^{(+)} \\ \vec{e}_2^{(+)} \end{pmatrix} = (\vec{e}_1^{(-)T}, \vec{a}_2^{(-)T}) \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} \vec{e}_1^{(+)} \\ \vec{a}_2^{(+)} \end{pmatrix}$$

$$\Rightarrow \tilde{T}^{(-)T} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \tilde{T}^{(+)} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

2. Step: Calculate the S-matrix or its inverse/transpose

$$S^{(\pm)} = \begin{pmatrix} S_{11}^{(\pm)} & S_{12}^{(\pm)} \\ S_{21}^{(\pm)} & S_{22}^{(\pm)} \end{pmatrix} = \begin{pmatrix} (\tilde{T}_{11}^{(\pm)} - \tilde{T}_{12}^{(\pm)} \tilde{T}_{22}^{(\pm)-1} \tilde{T}_{21}^{(\pm)}) & \tilde{T}_{12}^{(\pm)} \tilde{T}_{22}^{(\pm)-1} \\ -\tilde{T}_{22}^{(\pm)-1} \tilde{T}_{21}^{(\pm)} & \tilde{T}_{22}^{(\pm)-1} \end{pmatrix} = [S^{(\mp)-1}]^T$$

$$S^{(\pm)-1} = \begin{pmatrix} \tilde{T}_{11}^{(\pm)} & -\tilde{T}_{11}^{(\pm)-1} \tilde{T}_{12}^{(\pm)} \\ \tilde{T}_{21}^{(\pm)} \tilde{T}_{11}^{(\pm)} & (\tilde{T}_{22}^{(\pm)} - \tilde{T}_{21}^{(\pm)} \tilde{T}_{11}^{(\pm)-1} \tilde{T}_{12}^{(\pm)}) \end{pmatrix} = S^{(\mp)T}$$

$$S^{(\pm)T} = \begin{pmatrix} S_{11}^{(\pm)T} & S_{21}^{(\pm)T} \\ S_{12}^{(\pm)T} & S_{22}^{(\pm)T} \end{pmatrix} = \begin{pmatrix} (\tilde{T}_{11}^{(\pm)T} - \tilde{T}_{21}^{(\pm)T} \tilde{T}_{22}^{(\pm)-1^T} \tilde{T}_{12}^{(\pm)T}) & -\tilde{T}_{21}^{(\pm)T} \tilde{T}_{22}^{(\pm)-1T} \\ \tilde{T}_{22}^{(\pm)-1T} \tilde{T}_{12}^{(\pm)T} & \tilde{T}_{22}^{(\pm)-1T} \end{pmatrix}$$

$$[S^{(\pm)-1}]^T = \begin{pmatrix} \tilde{T}_{11}^{(\pm)-1T} & \tilde{T}_{11}^{(\pm)-1T} \tilde{T}_{21}^{(\pm)T} \\ -\tilde{T}_{12}^{(\pm)T} \tilde{T}_{11}^{(\pm)-1T} & (\tilde{T}_{22}^{(\pm)T} - \tilde{T}_{12}^{(\pm)T} \tilde{T}_{11}^{(\pm)-1^T} \tilde{T}_{21}^{(\pm)T}) \end{pmatrix} = S^{(\mp)}$$

3. Step: Calculate the transmittivities T_1, T_2 and reflectiviti[es] R_1, R_2

$$T_1 = 1 - R_1 = S_{11}^{(-)T} S_{11}^{(+)} = \tilde{T}_{11}^{(+)-1} \tilde{T}_{11}^{(-)T-1} = [\tilde{T}_{11}^{(-)T} \tilde{T}_{11}^{(+)}]^{-1}, \quad T_2 = 1 - R_2 = [\tilde{T}_{22}^{(+)} \tilde{T}_{22}^{(-)T}$$

Relation between $\tilde{T}^{(\pm)}$ and the transfermatrix $T^{(\pm)}$

Assumption: Interaction only for $z \in [z_<, z_>]$

Recall:

$$\begin{pmatrix} \underbrace{\sqrt{\tfrac{\hbar^2}{2M}}}_{\sqrt{\frac{\hbar}{2k_0}}\sqrt{\frac{\hbar k_0}{M}}} \vec{\Phi}^{(\pm)}(z_>) \\ \sqrt{\tfrac{\hbar^2}{2M}}\, \vec{\Phi}^{(\pm)}(z_>)' \end{pmatrix} = \underbrace{\begin{pmatrix} T_{11}^{(\pm)} & T_{12}^{(\pm)} \\ T_{21}^{(\pm)} & T_{22}^{(\pm)} \end{pmatrix}}_{T^{(\pm)}} \begin{pmatrix} \underbrace{\sqrt{\tfrac{\hbar^2}{2M}}}_{\sqrt{\frac{\hbar}{2k_0}}\sqrt{\frac{\hbar k_0}{M}}} \vec{\Phi}^{(\pm)}(z_<) \\ \sqrt{\tfrac{\hbar^2}{2M}}\, \vec{\Phi}^{(\pm)}(z_<)' \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ \pm ik_0 & \mp ik_0 \end{pmatrix} \sqrt{\tfrac{\hbar}{2k_0}} \begin{pmatrix} \vec{a}_1^{(\pm)} \\ \vec{e}_2^{(\pm)} \end{pmatrix} = T^{(\pm)} \begin{pmatrix} 1 & 1 \\ \pm ik_0 & \mp ik_0 \end{pmatrix} \sqrt{\tfrac{\hbar}{2k_0}} \begin{pmatrix} \vec{e}_1^{(\pm)} \\ \vec{a}_2^{(\pm)} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \vec{a}_1^{(\pm)} \\ \vec{e}_2^{(\pm)} \end{pmatrix} = \underbrace{\sqrt{k_0} \begin{pmatrix} 1 & 1 \\ \pm ik_0 & \mp ik_0 \end{pmatrix}^{-1} T^{(\pm)} \begin{pmatrix} 1 & 1 \\ \pm ik_0 & \mp ik_0 \end{pmatrix} \frac{1}{\sqrt{k_0}}}_{\tilde{T}^{(\pm)}} \begin{pmatrix} \vec{e}_1^{(\pm)} \\ \vec{a}_2^{(\pm)} \end{pmatrix}$$

$$\Rightarrow \boxed{\tilde{T}^{(\pm)} = \frac{1}{2}\sqrt{k_0} \begin{pmatrix} 1 & \pm\frac{1}{ik_0} \\ 1 & \mp\frac{1}{ik_0} \end{pmatrix} T^{(\pm)} \begin{pmatrix} 1 & 1 \\ \pm ik_0 & \mp ik_0 \end{pmatrix} \frac{1}{\sqrt{k_0}}}$$

$$\Rightarrow \tilde{T}_{11}^{(\pm)} = \frac{1}{2}\sqrt{k_0}\left(1, \pm\frac{1}{ik_0}\right) T^{(\pm)} \begin{pmatrix} 1 \\ \pm ik_0 \end{pmatrix} \frac{1}{\sqrt{k_0}}$$

$$\tilde{T}_{22}^{(\pm)} = \frac{1}{2}\sqrt{k_0}\left(1, \mp\frac{1}{ik_0}\right) T^{(\pm)} \begin{pmatrix} 1 \\ \mp ik_0 \end{pmatrix} \frac{1}{\sqrt{k_0}}$$

$$\tilde{T}_{11}^{(\pm)T} = \frac{1}{2}\frac{1}{\sqrt{k_0}}\left(1, \pm ik_0\right) T^{(\pm)T} \begin{pmatrix} 1 \\ \pm\frac{1}{ik_0} \end{pmatrix} \sqrt{k_0}$$

$$\tilde{T}_{22}^{(\pm)T} = \frac{1}{2}\frac{1}{\sqrt{k_0}}\left(1, \mp ik_0\right) T^{(\pm)T} \begin{pmatrix} 1 \\ \mp\frac{1}{ik_0} \end{pmatrix} \sqrt{k_0}$$

By construction there holds:

$$\boxed{\tilde{T}^{(\pm)} = \mathbb{1} + \tfrac{1}{2}\sqrt{k_0}\begin{pmatrix} 1 \pm \frac{1}{ik_0} \\ 1 \mp \frac{1}{ik_0}\end{pmatrix}\left(T^{(\pm)} - \mathbb{1}\right)\begin{pmatrix} 1 & 1 \\ \pm ik_0 & \mp ik_0\end{pmatrix}\frac{1}{\sqrt{k_0}}}$$

The transfer matrix $T^{(\pm)}$ is obtained by integration of the following two differential equations:

$$\frac{d}{dz}\begin{pmatrix}\sqrt{\frac{\hbar^2}{2M}}\,\vec{\Phi}^{(+)}(z) \\ \sqrt{\frac{\hbar^2}{2M}}\,\vec{\Phi}^{(+)}(z)'\end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ -\underbrace{\left[\frac{2M}{\hbar^2}E - U(z)\right]}_{k^2(z)} & 0\end{pmatrix}\begin{pmatrix}\sqrt{\frac{\hbar^2}{2M}}\,\vec{\Phi}^{(+)}(z) \\ \sqrt{\frac{\hbar^2}{2M}}\,\vec{\Phi}^{(+)}(z)'\end{pmatrix}$$

$$\frac{d}{dz}\begin{pmatrix}\sqrt{\frac{\hbar^2}{2M}}\,\vec{\Phi}^{(-)}(z) \\ \sqrt{\frac{\hbar^2}{2M}}\,\vec{\Phi}^{(-)}(z)'\end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ -\underbrace{\left[\frac{2M}{\hbar^2}E - U^T(z)\right]}_{[k^2(z)]^T} & 0\end{pmatrix}\begin{pmatrix}\sqrt{\frac{\hbar^2}{2M}}\,\vec{\Phi}^{(-)}(z) \\ \sqrt{\frac{\hbar^2}{2M}}\,\vec{\Phi}^{(-)}(z)'\end{pmatrix}$$

Scattering at a δ-potential $V(z) = \delta(z-a)\,g$

$$\Rightarrow \begin{pmatrix}\sqrt{\frac{\hbar^2}{2M}}\,\vec{\Phi}^{(+)}(a+0) \\ \sqrt{\frac{\hbar^2}{2M}}\,\vec{\Phi}^{(+)}(a+0)'\end{pmatrix} = \underbrace{\begin{pmatrix}\mathbb{1} & 0 \\ \sqrt{\frac{2M}{\hbar^2}}\,g\,\sqrt{\frac{2M}{\hbar^2}} & \mathbb{1}\end{pmatrix}}_{T^{(+)}}\begin{pmatrix}\sqrt{\frac{\hbar^2}{2M}}\,\vec{\Phi}^{(+)}(a-0) \\ \sqrt{\frac{\hbar^2}{2M}}\,\vec{\Phi}^{(+)}(a-0)'\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}\sqrt{\frac{\hbar^2}{2M}}\,\vec{\Phi}^{(-)}(a+0) \\ \sqrt{\frac{\hbar^2}{2M}}\,\vec{\Phi}^{(-)}(a+0)'\end{pmatrix} = \underbrace{\begin{pmatrix}\mathbb{1} & 0 \\ \sqrt{\frac{2M}{\hbar^2}}\,g^T\,\sqrt{\frac{2M}{\hbar^2}} & \mathbb{1}\end{pmatrix}}_{T^{(-)}}\begin{pmatrix}\sqrt{\frac{\hbar^2}{2M}}\,\vec{\Phi}^{(-)}(a-0) \\ \sqrt{\frac{\hbar^2}{2M}}\,\vec{\Phi}^{(-)}(a-0)'\end{pmatrix}$$

$$\Rightarrow \tilde{T}^{(+)} = \mathbb{1} + \tfrac{1}{2}\sqrt{k_0}\begin{pmatrix} 1 + \frac{1}{ik_0} \\ 1 - \frac{1}{ik_0}\end{pmatrix}\begin{pmatrix} 0 & 0 \\ \sqrt{\frac{2M}{\hbar^2}}\,g\,\sqrt{\frac{2M}{\hbar^2}} & 0\end{pmatrix}\begin{pmatrix} 1 & 1 \\ +ik_0 & -ik_0\end{pmatrix}\frac{1}{\sqrt{k_0}}$$

$$\tilde{T}^{(-)} = \mathbb{1} + \tfrac{1}{2}\sqrt{k_0}\begin{pmatrix} 1 - \frac{1}{ik_0} \\ 1 + \frac{1}{ik_0}\end{pmatrix}\begin{pmatrix} 0 & 0 \\ \sqrt{\frac{2M}{\hbar^2}}\,g^T\sqrt{\frac{2M}{\hbar^2}} & 0\end{pmatrix}\begin{pmatrix} 1 & 1 \\ -ik_0 & +ik_0\end{pmatrix}\frac{1}{\sqrt{k_0}}$$

Scattering at a δ-potential $V(z)=\delta(z-a)g$ continued

$$\Rightarrow \tilde{T}_{11}^{(+)} = \mathbb{1} + \sqrt{k_0}\,\frac{1}{2ik_0}\sqrt{\frac{2M}{\hbar^2}}\,g\,\sqrt{\frac{2M}{\hbar^2}}\,\frac{1}{\sqrt{k_0}} = \left[S_{11}^{(-)T}\right]^{-1}$$

$$\tilde{T}_{22}^{(+)} = \mathbb{1} - \sqrt{k_0}\,\frac{1}{2ik_0}\sqrt{\frac{2M}{\hbar^2}}\,g\,\sqrt{\frac{2M}{\hbar^2}}\,\frac{1}{\sqrt{k_0}} = \left[S_{22}^{(+)}\right]^{-1}$$

$$\tilde{T}_{11}^{(-)} = \mathbb{1} - \sqrt{k_0}\,\frac{1}{2ik_0}\sqrt{\frac{2M}{\hbar^2}}\,g^T\sqrt{\frac{2M}{\hbar^2}}\,\frac{1}{\sqrt{k_0}}$$

$$\tilde{T}_{22}^{(-)} = \mathbb{1} + \sqrt{k_0}\,\frac{1}{2ik_0}\sqrt{\frac{2M}{\hbar^2}}\,g^T\sqrt{\frac{2M}{\hbar^2}}\,\frac{1}{\sqrt{k_0}}$$

$$\Rightarrow \tilde{T}_{11}^{(-)T} = \mathbb{1} - \frac{1}{\sqrt{k_0}}\sqrt{\frac{2M}{\hbar^2}}\,g\,\sqrt{\frac{2M}{\hbar^2}}\,\frac{1}{2ik_0}\sqrt{k_0} = \left[S_{11}^{(+)}\right]^{-1}$$

$$\tilde{T}_{22}^{(-)T} = \mathbb{1} + \frac{1}{\sqrt{k_0}}\sqrt{\frac{2M}{\hbar^2}}\,g\,\sqrt{\frac{2M}{\hbar^2}}\,\frac{1}{2ik_0}\sqrt{k_0} = \left[S_{22}^{(-)T}\right]^{-1}$$

Recall:

$$\boxed{\begin{aligned} T_1 &= 1 - R_1 = S_{11}^{(-)T} S_{11}^{(+)} = \tilde{T}_{11}^{(+)-1}\,T_{11}^{(-)T-1} = \left[\tilde{T}_{11}^{(-)T} T_{11}^{(+)}\right]^{-1} \\ T_2 &= 1 - R_2 = S_{22}^{(-)T} S_{22}^{(+)} = T_{22}^{(-)T-1}\,T_{22}^{(+)-1} = \left[T_{22}^{(+)} T_{22}^{(-)T}\right]^{-1} \end{aligned}}$$

$$\Rightarrow T_1 = 1 - R_1 = \left[\left(\mathbb{1} - \frac{1}{2i}\frac{1}{\sqrt{k_0}}\sqrt{\frac{2M}{\hbar^2}}\,g\,\sqrt{\frac{2M}{\hbar^2}}\,\frac{1}{\sqrt{k_0}}\right)\left(\mathbb{1} + \frac{1}{2i}\frac{1}{\sqrt{k_0}}\sqrt{\frac{2M}{\hbar^2}}\,g\,\sqrt{\frac{2M}{\hbar^2}}\,\frac{1}{\sqrt{k_0}}\right)\right]^{-1}$$

$$T_2 = 1 - R_2 = \left[\left(\mathbb{1} - \frac{1}{2i}\frac{1}{\sqrt{k_0}}\sqrt{\frac{2M}{\hbar^2}}\,g\,\sqrt{\frac{2M}{\hbar^2}}\,\frac{1}{\sqrt{k_0}}\right)\left(\mathbb{1} + \frac{1}{2i}\frac{1}{\sqrt{k_0}}\sqrt{\frac{2M}{\hbar^2}}\,g\,\sqrt{\frac{2M}{\hbar^2}}\,\frac{1}{\sqrt{k_0}}\right)\right]^{-1}$$

One scattering channel:

$$\Rightarrow \left.\begin{aligned} T_1 &= 1 - R_1 \\ T_2 &= 1 - R_2 \end{aligned}\right\} = \frac{1}{\left(1 - \frac{1}{2ik_0}\frac{2M}{\hbar^2}g\right)\left(1 + \frac{1}{2ik_0}\frac{2M}{\hbar^2}g\right)} = \underbrace{\frac{1}{1 + \frac{1}{4k_0^2}\left(\frac{2M}{\hbar^2}\right)^2 g^2}}_{\displaystyle\frac{1}{1 + \frac{1}{4E}\frac{2M}{\hbar^2}g^2}}$$

$$\boxed{Mg^2 \in \mathbb{R}_0^+} \Leftarrow$$

Scattering at some constant potential

$$V(z) = V = \begin{pmatrix} V_{11} \dots V_{1n} \\ \vdots \\ V_{n1} \dots V_{nn} \end{pmatrix} = \text{konst} \neq 0 \quad \text{for } z \in [z_<, z_>]$$

$$\Rightarrow k^2 = \frac{2M}{\hbar^2} E - \sqrt{\frac{2M}{\hbar^2}} V \sqrt{\frac{2M}{\hbar^2}} = \frac{2M}{\hbar^2} E - U = \text{konst for } z \in [z_<, z_>]$$

k^2 can be diagonalised: $k^2 = \underbrace{X}_{\text{matrix of right eigenvectors}} \overbrace{[k^2]_D}^{\text{diagonal matrix}} X^{-1}$

Now we take the squareroot of a matrix:

$$k = X \underbrace{\sqrt{[k^2]_D}}_{k_D} X^{-1}$$

k_D is ambiguous $\Rightarrow \exists\ 2^n$ possibilities

$\Rightarrow$ Riemann-sheet structure of scattering plane?

Transfermatrix $T^{(\pm)} = T^{(\pm)}(z_>, z_<)$: $\quad a := z_> - z_<$

$$T^{(+)} = \begin{pmatrix} \cos(ka) & \frac{\sin(ka)}{k} \\ -k\sin(ka) & \cos(ka) \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} \cos(k_D a) & \frac{\sin(k_D a)}{k_D} \\ -k_D \sin(k_D a) & \cos(k_D a) \end{pmatrix} \begin{pmatrix} X^{-1} & 0 \\ 0 & X^{-1} \end{pmatrix}$$

$$T^{(-)} = \begin{pmatrix} \cos(k^T a) & \frac{\sin(k^T a)}{k^T} \\ -k^T \sin(k^T a) & \cos(k^T a) \end{pmatrix} = \begin{pmatrix} X^{-1T} & 0 \\ 0 & X^{-1T} \end{pmatrix} \underbrace{\begin{pmatrix} \cos(k_D a) & \frac{\sin(k_D a)}{k_D} \\ -k_D \sin(k_D a) & \cos(k_D a) \end{pmatrix}}_{T_D(a)} \begin{pmatrix} X^T & 0 \\ 0 & X^T \end{pmatrix}$$

Scattering at some constant potential (continued) 17a

Diskussion on the level of the wavefunction

$$\vec{\Phi}^{(+)}(z_>) = \cos(k(z_>-z_<))\,\vec{\Phi}^{(+)}(z_<) + \frac{\sin(k(z_>-z_<))}{k}\,\vec{\Phi}^{(+)}(z_<)'$$

$$\vec{\Phi}^{(-)}(z_>) = \cos(k^T(z_>-z_<))\,\vec{\Phi}^{(-)}(z_<) + \frac{\sin(k^T(z_>-z_<))}{k^T}\,\vec{\Phi}^{(-)}(z_<)'$$

$$\Rightarrow \vec{\Phi}^{(+)}(z_>) = e^{ik(z_>-z_<)}\,\frac{1}{2}\Big(\vec{\Phi}^{(+)}(z_<) + \frac{1}{ik}\vec{\Phi}^{(+)}(z_<)'\Big)$$

$$+\, e^{-ik(z_>-z_<)}\,\frac{1}{2}\Big(\vec{\Phi}^{(+)}(z_<) - \frac{1}{ik}\vec{\Phi}^{(+)}(z_<)'\Big)$$

$$= X\Big[e^{ik_D(z_>-z_<)}\,\frac{1}{2}\Big(X^{-1}\vec{\Phi}^{(+)}(z_<) + \frac{1}{ik_D}X^{-1}\vec{\Phi}^{(+)}(z_<)'\Big)$$

$$+\, e^{-ik_D(z_>-z_<)}\,\frac{1}{2}\Big(X^{-1}\vec{\Phi}^{(+)}(z_<) - \frac{1}{ik_D}X^{-1}\vec{\Phi}^{(+)}(z_<)'\Big)\Big]$$

$$\vec{\Phi}^{(-)}(z_>) = e^{ik^T(z_>-z_<)}\,\frac{1}{2}\Big(\vec{\Phi}^{(-)}(z_<) + \frac{1}{ik^T}\vec{\Phi}^{(-)}(z_<)'\Big)$$

$$+\, e^{-ik^T(z_>-z_<)}\,\frac{1}{2}\Big(\vec{\Phi}^{(-)}(z_<) - \frac{1}{ik^T}\vec{\Phi}^{(-)}(z_<)'\Big)$$

$$= X^{-1T}\Big[e^{ik_D(z_>-z_<)}\,\frac{1}{2}\Big(X^T\vec{\Phi}^{(-)}(z_<) + \frac{1}{ik_D}X^T\vec{\Phi}^{(-)}(z_<)'\Big)$$

$$+\, e^{-ik_D(z_>-z_<)}\,\frac{1}{2}\Big(X^T\vec{\Phi}^{(-)}(z_<) - \frac{1}{ik_D}X^T\vec{\Phi}^{(-)}(z_<)'\Big)\Big]$$

Recall $e^{\pm ik_D(z_>-z_<)} = e^{\pm i(\mathrm{Re}\,k_D + i\,\mathrm{Im}\,k_D)(z_>-z_<)}$

$$= \underbrace{e^{\pm i\,\mathrm{Re}\,k_D\cdot(z_>-z_<)}}_{\text{oscillation}}\;\underbrace{e^{\mp\mathrm{Im}\,k_D\cdot(z_>-z_<)}}_{\text{exp.}}$$

exp. → decrease
↘ increase

Scattering at some constant potential (continued)

Recall: $K = X\sqrt{k_D^2}\,X^{-1} = X\, k_D\, X^{-1}$ with $k_D = k_R + i k_I$

$$T_D(a) = \begin{pmatrix} \cos(k_D a) & \frac{\sin(k_D a)}{k_D} \\ -k_D \sin(k_D a) & \cos(k_D a) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & k_D \end{pmatrix} \underbrace{\begin{pmatrix} \cosh(k_I a) & i\sinh(k_I a) \\ -i\sinh(k_I a) & \cosh(k_I a) \end{pmatrix}}_{\xrightarrow{a\to\infty} \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} e^{k_I a}} \begin{pmatrix} \cos(k_R a) & \sin(k_R a) \\ -\sin(k_R a) & \cos(k_R a) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{k_D} \end{pmatrix}$$

There holds:

$$\tilde{T}^{(+)} = \frac{1}{2}\sqrt{k_0} \begin{pmatrix} 1 & \frac{1}{ik_0} \\ 1 & -\frac{1}{ik_0} \end{pmatrix} \underbrace{\begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} T_D(a) \begin{pmatrix} X^{-1} & 0 \\ 0 & X^{-1} \end{pmatrix}}_{T^{(+)}} \begin{pmatrix} 1 & 1 \\ +ik_0 & -ik_0 \end{pmatrix} \frac{1}{\sqrt{k_0}}$$

$$\tilde{T}^{(-)} = \frac{1}{2}\sqrt{k_0} \begin{pmatrix} 1 & -\frac{1}{ik_0} \\ 1 & \frac{1}{ik_0} \end{pmatrix} \begin{pmatrix} X^{-1T} & 0 \\ 0 & X^{-1T} \end{pmatrix} T_D(a) \begin{pmatrix} X^T & 0 \\ 0 & X^T \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -ik_0 & +ik_0 \end{pmatrix} \frac{1}{\sqrt{k_0}}$$

$$\Rightarrow \tilde{T}^{(+)}_{11} = \frac{1}{2}\sqrt{k_0}\left(X, \frac{1}{ik_0}X\right) T_D(a) \begin{pmatrix} X^{-1} \\ X^{-1} ik_0 \end{pmatrix} \frac{1}{\sqrt{k_0}}$$

$$\tilde{T}^{(+)}_{22} = \frac{1}{2}\sqrt{k_0}\left(X, -\frac{1}{ik_0}X\right) T_D(a) \begin{pmatrix} X^{-1} \\ X^{-1}(-ik_0) \end{pmatrix} \frac{1}{\sqrt{k_0}}$$

$$\tilde{T}^{(-)}_{11} = \frac{1}{2}\sqrt{k_0}\left(X^{-1T}, -\frac{1}{ik_0}X^{-1T}\right) T_D(a) \begin{pmatrix} X^{T} \\ X^{T}(-ik_0) \end{pmatrix} \frac{1}{\sqrt{k_0}}$$

$$\tilde{T}^{(-)}_{22} = \frac{1}{2}\sqrt{k_0}\left(X^{-1T}, \frac{1}{ik_0}X^{-1T}\right) T_D(a) \begin{pmatrix} X^{T} \\ X^{T} ik_0 \end{pmatrix} \frac{1}{\sqrt{k_0}}$$

$$\Rightarrow \tilde{T}^{(-)T}_{11} = \frac{1}{2}\frac{1}{\sqrt{k_0}}\left(X, -ik_0 X\right) T_D(a) \begin{pmatrix} X^{-1} \\ X^{-1}\left(-\frac{1}{ik_0}\right) \end{pmatrix} \sqrt{k_0}$$

$$\tilde{T}^{(-)T}_{22} = \frac{1}{2}\frac{1}{\sqrt{k_0}}\left(X, ik_0 X\right) T_D(a) \begin{pmatrix} X^{-1} \\ X^{-1}\frac{1}{ik_0} \end{pmatrix} \sqrt{k_0}$$

Scattering at constant potential (continued)

$$T_1 = [\tilde{T}_{11}^{(-)T}\tilde{T}_{11}^{(+)}]^{-1} = 1 - R_1 =$$

$$= \Big[\Big(1 + \frac{1}{2}\frac{1}{\sqrt{k_0}}(X, -ik_0X)(T_D(a) - 1)\begin{pmatrix} X^{-1} \\ X^{-1}(-\frac{1}{ik_0})\end{pmatrix}\sqrt{k_0}\Big)$$

$$\cdot\Big(1 + \frac{1}{2}\sqrt{k_0}(X, \frac{1}{ik_0}X)(T_D(a) - 1)\begin{pmatrix} X^{-1} \\ X^{-1}\,ik_0\end{pmatrix}\frac{1}{\sqrt{k_0}}\Big)\Big]^{-1}$$

$$T_2 = [\tilde{T}_{22}^{(+)}\tilde{T}_{22}^{(-)T}]^{-1} = 1 - R_2 =$$

$$= \Big[\Big(1 + \frac{1}{2}\sqrt{k_0}(X, -\frac{1}{ik_0}X)(T_D(a) - 1)\begin{pmatrix} X^{-1} \\ X^{-1}(-ik_0)\end{pmatrix}\frac{1}{\sqrt{k_0}}\Big)$$

$$\Big(1 + \frac{1}{2}\frac{1}{\sqrt{k_0}}(X, ik_0X)(T_D(a) - 1)\begin{pmatrix} X^{-1} \\ X^{-1}\frac{1}{ik_0}\end{pmatrix}\sqrt{k_0}\Big)\Big]^{-1}$$

One scattering channel:

$$T_1 = 1 - R_1 = \frac{4}{(1, -ik_0)\begin{pmatrix}\cos(k_Da) & \frac{\sin(k_Da)}{k_D} \\ -k_D\sin(k_Da) & \cos(k_Da)\end{pmatrix}\begin{pmatrix} 1 & \frac{1}{ik_0} \\ -\frac{1}{ik_0} & \frac{1}{k_0^2}\end{pmatrix}\begin{pmatrix}\cos(k_Da) & \frac{\sin(k_Da)}{k_D} \\ -k_D\sin(k_Da) & \cos(k_Da)\end{pmatrix}\begin{pmatrix}1 \\ ik_0\end{pmatrix}}$$

$$T_2 = 1 - R_2 = \frac{4}{(1, -\frac{1}{ik_0})\begin{pmatrix}\cos(k_Da) & \frac{\sin(k_Da)}{k_D} \\ -k_D\sin(k_Da) & \cos(k_Da)\end{pmatrix}\begin{pmatrix} 1 & ik_0 \\ -ik_0 & k_0^2\end{pmatrix}\begin{pmatrix}\cos(k_Da) & \frac{\sin(k_Da)}{k_D} \\ -k_D\sin(k_Da) & \cos(k_Da)\end{pmatrix}\begin{pmatrix}1 \\ \frac{1}{ik_0}\end{pmatrix}}$$

Scattering at a constant potential (continued)

TWO scattering channels:

$$k^2 = \begin{pmatrix} \left[\frac{2m_1}{\hbar^2}E - U_{11}\right] & -U_{12} \\ -U_{21} & \left[\frac{2m_2}{\hbar^2}E - U_{22}\right] \end{pmatrix} \xrightarrow[\substack{U_{11}=U_{22}=0 \\ m_1=m_2=m}]{\text{for simplicity}} \begin{pmatrix} \frac{2m}{\hbar^2}E & -U_{12} \\ -U_{21} & \frac{2m}{\hbar^2}E \end{pmatrix}$$

$$\Rightarrow k^2 = \begin{pmatrix} \frac{2m}{\hbar^2}E & -U_{12} \\ -U_{21} & \frac{2m}{\hbar^2}E \end{pmatrix}$$

$$= \underbrace{\frac{1}{\sqrt{2}\sqrt[4]{U_{12}U_{21}}}\begin{pmatrix} \sqrt{U_{12}} & \sqrt{U_{12}} \\ -\sqrt{U_{21}} & \sqrt{U_{21}} \end{pmatrix}}_{X} \underbrace{\begin{pmatrix} \left[\frac{2m}{\hbar^2}E + \sqrt{U_{12}U_{21}}\right] & 0 \\ 0 & \left[\frac{2m}{\hbar^2}E - \sqrt{U_{12}U_{21}}\right] \end{pmatrix}}_{[k^2]_D} \underbrace{\begin{pmatrix} \sqrt{U_{21}} & -\sqrt{U_{12}} \\ \sqrt{U_{21}} & \sqrt{U_{12}} \end{pmatrix} \cdot \frac{1}{\sqrt{2}\sqrt[4]{U_{12}U_{21}}}}_{X^{-1}}$$

$$\Rightarrow \quad k = X \underbrace{\begin{pmatrix} \pm\overbrace{\sqrt{\frac{2m}{\hbar^2}E + \sqrt{U_{12}U_{21}}}}^{k_+} & 0 \\ 0 & \pm\underbrace{\sqrt{\frac{2m}{\hbar^2}E - \sqrt{U_{12}U_{21}}}}_{k_-} \end{pmatrix}}_{k_D = \begin{pmatrix} k_+ & 0 \\ 0 & k_- \end{pmatrix}} X^{-1}$$

Transfer matrix: $\quad T^{(+)} = \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} \cos(k_D a) & \frac{\sin(k_D a)}{k_D} \\ -k_D \sin(k_D a) & \cos(k_D a) \end{pmatrix} \begin{pmatrix} X^{-1} & 0 \\ 0 & X^{-1} \end{pmatrix}$

$(a = z_> - z_<)$

$$T^{(-)} = \begin{pmatrix} X^{-1T} & 0 \\ 0 & X^{-1T} \end{pmatrix} \begin{pmatrix} \cos(k_D a) & \frac{\sin(k_D a)}{k_D} \\ -k_D \sin(k_D a) & \cos(k_D a) \end{pmatrix} \begin{pmatrix} X^T & 0 \\ 0 & X^T \end{pmatrix}$$

Effect of X on diagonal matrix:

$$X \begin{pmatrix} d_+ & 0 \\ 0 & d_- \end{pmatrix} X^{-1} = \frac{1}{2} \begin{pmatrix} (d_- + d_+) & (d_- - d_+) \cdot \sqrt{\frac{U_{12}}{U_{21}}} \\ (d_- - d_+) \cdot \sqrt{\frac{U_{21}}{U_{12}}} & (d_- + d_+) \end{pmatrix}$$

Scattering at a double-δ-potential:

$$V(z) = g_> \delta(z-a_>) + g_< \delta(z-a_<) \quad (a = a_> - a_<)$$

Transfer matrix: $T^{(\pm)} = T_>^{(\pm)}(a) T_0(a) T_<^{(\pm)}(a)$

$$T_>^{(+)} = \begin{pmatrix} 1 & 0 \\ \sqrt{\frac{2M}{\hbar^2}} g_> \sqrt{\frac{2M}{\hbar^2}} & 1 \end{pmatrix}, \quad T_<^{(+)} = \begin{pmatrix} 1 & 0 \\ \sqrt{\frac{2M}{\hbar^2}} g_< \sqrt{\frac{2M}{\hbar^2}} & 1 \end{pmatrix}$$

$$T_>^{(-)} = \begin{pmatrix} 1 & 0 \\ \sqrt{\frac{2M}{\hbar^2}} g_>^T \sqrt{\frac{2M}{\hbar^2}} & 1 \end{pmatrix}, \quad T_<^{(-)} = \begin{pmatrix} 1 & 0 \\ \sqrt{\frac{2M}{\hbar^2}} g_<^T \sqrt{\frac{2M}{\hbar^2}} & 1 \end{pmatrix}$$

$$T_0(a) = \begin{pmatrix} \cos(k_0 a) & \frac{\sin(k_0 a)}{k_0} \\ -k_0 \sin(k_0 a) & \cos(k_0 a) \end{pmatrix}$$

Recall:
$$\begin{aligned} T^{(\pm)} &= [1 - (T_>^{(\pm)} - 1)] T_0(a) [1 - (T_<^{(\pm)} - 1)] \\ &= T_0(a) + (T_>^{(\pm)} - 1) T_0(a) + T_0(a)(T_<^{(\pm)} - 1) \\ &\quad + (T_>^{(\pm)} - 1) T_0(a) (T_<^{(\pm)} - 1) \end{aligned}$$

and
$$\tilde{T}^{(\pm)} = \frac{1}{2}\sqrt{k_0} \begin{pmatrix} 1 \pm \frac{1}{ik_0} \\ 1 \mp \frac{1}{ik_0} \end{pmatrix} T^{(\pm)} \begin{pmatrix} 1 & 1 \\ \pm ik_0 & \mp ik_0 \end{pmatrix} \frac{1}{\sqrt{k_0}}$$

Important feature: $T_0(a) \begin{pmatrix} 1 \\ \pm ik_0 \end{pmatrix} = e^{\pm ik_0 a} \begin{pmatrix} 1 \\ \pm ik_0 \end{pmatrix}$

$$\left(1, \pm \frac{1}{ik_0}\right) T_0(a) = \left(1, \pm \frac{1}{ik_0}\right) e^{\pm ik_0 a}$$

Result:
$$\begin{aligned} \tilde{T}_{11}^{(\pm)} &= \frac{1}{\sqrt{2}}\sqrt{k_0}\left(1 \pm \frac{1}{ik_0}\right)\Big[e^{\pm ik_0 a} + e^{\pm ik_0 a}(T_<^{(\pm)} - 1) + (T_>^{(\pm)} - 1)e^{\pm ik_0 a} \\ &\quad + (T_>^{(\pm)} - 1) T_0(a) (T_<^{(\pm)} - 1)\Big] \begin{pmatrix} 1 \\ \pm ik_0 \end{pmatrix} \frac{1}{\sqrt{k_0}} \end{aligned}$$

$$\begin{aligned} \tilde{T}_{22}^{(\pm)} &= \frac{1}{\sqrt{2}}\sqrt{k_0}\left(1 \mp \frac{1}{ik_0}\right)\Big[e^{\mp ik_0 a} + e^{\mp ik_0 a}(T_<^{(\pm)} - 1) + (T_>^{(\pm)} - 1)e^{\mp ik_0 a} \\ &\quad + (T_>^{(\pm)} - 1) T_0(a) (T_<^{(\pm)} - 1)\Big] \begin{pmatrix} 1 \\ \mp ik_0 \end{pmatrix} \frac{1}{\sqrt{k_0}} \end{aligned}$$

Scattering at a double-δ-potential (continued)

Recall:

$$(T_{\gtrless}^{(+)}-1)=\begin{pmatrix}0 & 0\\ \sqrt{\frac{2\mu}{\hbar^2}}\,g_{\gtrless}\sqrt{\frac{2\mu}{\hbar^2}} & 0\end{pmatrix},\quad (T_{\gtrless}^{(-)}-1)^T=\begin{pmatrix}0 & \sqrt{\frac{2\mu}{\hbar^2}}\,g_{\gtrless}\sqrt{\frac{2\mu}{\hbar^2}}\\ 0 & 0\end{pmatrix}$$

$$\rightsquigarrow T_1=\mathbb{1}-R_1=[\tilde{T}_{11}^{(-)T}\tilde{T}_{11}^{(+)}]^{-1}$$

$$=\Big[\Big(\mathbb{1}+\frac{1}{2}\frac{1}{\sqrt{k_0}}(1,-ik_0)\big[e^{-ik_0a}(T_>^{(-)}-1)^T+(T_<^{(-)}-1)^Te^{-ik_0a}$$
$$+(T_<^{(-)}-1)^TT_0(a)^T(T_>^{(-)}-1)^T\big]\begin{pmatrix}1\\-\frac{1}{ik_0}\end{pmatrix}\sqrt{k_0}\,e^{ik_0a}\Big)$$

$$\Big(\mathbb{1}+\frac{1}{2}e^{-ik_0a}\sqrt{k_0}\Big(1,\frac{1}{ik_0}\Big)\big[(T_>^{(+)}-1)e^{ik_0a}+e^{ik_0a}(T_<^{(+)}-1)$$
$$+(T_>^{(+)}-1)T_0(a)(T_<^{(+)}-1)\big]\begin{pmatrix}1\\ik_0\end{pmatrix}\frac{1}{\sqrt{k_0}}\Big)\Big]^{-1}$$

$$T_2=\mathbb{1}-R_2=[\tilde{T}_{22}^{(+)}\tilde{T}_{22}^{(-)T}]^{-1}$$

$$=\Big[\Big(\mathbb{1}+\frac{1}{2}\sqrt{k_0}\Big(1,-\frac{1}{ik_0}\Big)\big[(T_>^{(+)}-1)e^{-ik_0a}+e^{-ik_0a}(T_<^{(+)}-1)$$
$$+(T_>^{(+)}-1)T_0(a)(T_<^{(+)}-1)\big]\begin{pmatrix}1\\-ik_0\end{pmatrix}\frac{1}{\sqrt{k_0}}\,e^{ik_0a}\Big)$$

$$\Big(\mathbb{1}+\frac{1}{2}e^{-ik_0a}\frac{1}{\sqrt{k_0}}(1,ik_0)\big[e^{ik_0a}(T_>^{(-)}-1)^T+(T_<^{(-)}-1)^Te^{ik_0a}$$
$$+(T_<^{(-)}-1)^TT_0(a)^T(T_>^{(-)}-1)^T\big]\begin{pmatrix}1\\\frac{1}{ik_0}\end{pmatrix}\sqrt{k_0}\Big)\Big]^{-1}$$

Obviously the nonhermitian PT symmetric double-δ-potential yields a senseful theory:

$$V(z)=\underbrace{ig}_{g_>}\,\delta(z-a_>)\underbrace{-ig}_{g_<}\,\delta(z-a_<)\quad (g\in\mathbb{R})$$

(1 scattering channel!)

On the „unitarisation" including „bound states"

Starting point:
$$E\vec{\Phi}^{(+)}(z) = \left[-\frac{\hbar^2}{2M}\frac{d^2}{dz^2} + V(z)\right]\vec{\Phi}^{(+)}(z)$$
$$E\vec{\Phi}^{(-)}(z) = \left[-\frac{\hbar^2}{2M}\frac{d^2}{dz^2} + V(z)^T\right]\vec{\Phi}^{(-)}(z)$$

or, equivalently,
$$\frac{2M}{\hbar^2}E\sqrt{\frac{\hbar^2}{2M}}\vec{\Phi}^{(+)}(z) = \Big[-\frac{d^2}{dz^2} + \overbrace{\sqrt{\frac{2M}{\hbar^2}}V(z)\sqrt{\frac{2M}{\hbar^2}}}^{U(z)}\Big]\sqrt{\frac{\hbar^2}{2M}}\vec{\Phi}^{(+)}(z)$$
$$\frac{2M}{\hbar^2}E\sqrt{\frac{\hbar^2}{2M}}\vec{\Phi}^{(-)}(z) = \Big[-\frac{d^2}{dz^2} + \underbrace{\sqrt{\frac{2M}{\hbar^2}}V(z)^T\sqrt{\frac{2M}{\hbar^2}}}_{U(z)^T}\Big]\sqrt{\frac{\hbar^2}{2M}}\vec{\Phi}^{(-)}(z)$$

Switch to coupled channel model of scattering and confining („bound") states:

$$\vec{\Phi}^{(\pm)} = \begin{pmatrix}\vec{\Phi}_S^{(\pm)}\\ \vec{\Phi}_B^{(\pm)}\end{pmatrix} \qquad U(z) = \begin{pmatrix}U_{SS}(z) & U_{SB}(z)\\ U_{BS}(z) & U_{BB}(z)\end{pmatrix}$$

$$M = \begin{pmatrix}M_S & 0\\ 0 & M_B\end{pmatrix} \qquad = \begin{pmatrix}\sqrt{\frac{2M_S}{\hbar^2}}V_{SS}(z)\sqrt{\frac{2M_S}{\hbar^2}} & \sqrt{\frac{2M_S}{\hbar^2}}V_{SB}(z)\sqrt{\frac{2M_B}{\hbar^2}}\\ \sqrt{\frac{2M_B}{\hbar^2}}V_{BS}(z)\sqrt{\frac{2M_S}{\hbar^2}} & \sqrt{\frac{2M}{\hbar^2}}V_{BB}(z)\sqrt{\frac{2M_B}{\hbar^2}}\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}\frac{2M_S}{\hbar^2}E & 0\\ 0 & \frac{2M_B}{\hbar^2}E\end{pmatrix}\begin{pmatrix}\sqrt{\frac{\hbar^2}{2M_S}}\vec{\Phi}_S^{(+)}(z)\\ \sqrt{\frac{\hbar^2}{2M_B}}\vec{\Phi}_B^{(+)}(z)\end{pmatrix} = \begin{pmatrix}\left[-\frac{d^2}{dz^2}+U_{SS}\right] & U_{SB}\\ U_{BS} & \left[-\frac{d^2}{dz^2}+U_{BB}\right]\end{pmatrix}\begin{pmatrix}\sqrt{\frac{\hbar^2}{2M_S}}\vec{\Phi}_S^{(+)}(z)\\ \sqrt{\frac{\hbar^2}{2M_B}}\vec{\Phi}_B^{(+)}(z)\end{pmatrix}$$

$$\begin{pmatrix}\frac{2M_S}{\hbar^2}E & 0\\ 0 & \frac{2M}{\hbar^2}E\end{pmatrix}\begin{pmatrix}\sqrt{\frac{\hbar^2}{2M_S}}\vec{\Phi}_S^{(-)}(z)\\ \sqrt{\frac{\hbar^2}{2M_B}}\vec{\Phi}_B^{(-)}(z)\end{pmatrix} = \begin{pmatrix}\left[-\frac{d^2}{dz^2}+U_{SS}^T\right] & U_{BS}^T\\ U_{SB}^T & \left[-\frac{d^2}{dz^2}+U_{BB}^T\right]\end{pmatrix}\begin{pmatrix}\sqrt{\frac{\hbar^2}{2M_S}}\vec{\Phi}_S^{(-)}(z)\\ \sqrt{\frac{\hbar^2}{2M_B}}\vec{\Phi}_B^{(-)}(z)\end{pmatrix}$$

U_{SB} and U_{BS} are called <u>transition potentials</u> relating „bound" (B) and „scattering" (S) states.

The confining wavefunction $\vec{\Phi}_B^{(\pm)}(z)$ can be expanded in terms of eigensolutions $\vec{\Phi}_{Bn}^{(\pm)}(z)$ of the confining Hamiltonian, i.e.

$$\vec{\Phi}_B^{(\pm)}(z) = \sum_n c_{Bn}^{(\pm)} \vec{\Phi}_{Bn}^{(\pm)}(z)$$

with
$$\left[-\frac{d^2}{dz^2} + U_{BB}\right]\sqrt{\frac{\hbar^2}{2M_B}}\,\vec{\Phi}_{Bn}^{(+)}(z) = E_n \sqrt{\frac{\hbar^2}{2M_B}}\,\vec{\Phi}_{Bn}^{(+)}(z)$$
$$\left[-\frac{d^2}{dz^2} + U_{BB}^T\right]\sqrt{\frac{\hbar^2}{2M_B}}\,\vec{\Phi}_{Bn}^{(-)}(z) = E_n \sqrt{\frac{\hbar^2}{2M_B}}\,\vec{\Phi}_{Bn}^{(-)}(z)$$

and
$$\int_{z_1}^{z_2} dz' \; \vec{\Phi}_{Bn}^{(-)T}(z') \cdot \vec{\Phi}_{Bm}^{(+)}(z') = \delta_{mn}$$

$$\Rightarrow \sum_n \frac{2M_B}{\hbar^2}(E-E_n)\, c_{Bn}^{(+)} \sqrt{\frac{\hbar^2}{2M_B}}\,\vec{\Phi}_{Bn}^{(+)}(z) = U_{BS}\sqrt{\frac{\hbar^2}{2M_S}}\,\vec{\Phi}_S^{(+)}(z)$$
$$\sum_n \frac{2M_B}{\hbar^2}(E-E_n)\, c_{Bn}^{(-)} \sqrt{\frac{\hbar^2}{2M_B}}\,\vec{\Phi}_B^{(-)}(z) = U_{SB}^T\sqrt{\frac{\hbar^2}{2M_S}}\,\vec{\Phi}_S^{(-)}(z)$$

$$\Rightarrow c_{Bn}^{(+)} = \int_{z_1}^{z_2} dz' \, \frac{\vec{\Phi}_{Bn}^{(-)}(z')^T \cdot V_{BS}(z')\, \vec{\Phi}_S^{(+)}(z')}{E-E_n}$$
$$c_{Bn}^{(-)} = \int_{z_1}^{z_2} dz' \, \frac{\vec{\Phi}_{Bn}^{(+)}(z')^T\, V_{SB}^T(z')\, \vec{\Phi}_S^{(-)}(z')}{E-E_n}$$

upto homogeneous terms!

$$\Rightarrow \vec{\Phi}_B^{(+)}(z) = \sum_n \vec{\Phi}_{Bn}^{(+)}(z) \int_{z_1}^{z_2} dz' \, \frac{\vec{\Phi}_{Bn}^{(-)}(z')^T\, V_{BS}(z')\, \vec{\Phi}_S^{(+)}(z')}{E-E_n}$$
$$\vec{\Phi}_B^{(-)}(z) = \sum_n \vec{\Phi}_{Bn}^{(-)}(z) \int_{z_1}^{z_2} dz' \, \frac{\vec{\Phi}_{Bn}^{(+)}(z')^T\, V_{SB}^T(z')\, \vec{\Phi}_S^{(-)}(z')}{E-E_n}$$

Recall the definition of the currents: ㉕

$$j(z) = \frac{1}{i\hbar}\left[\vec{\Phi}^{(-)}(z)^T \frac{\hbar^2}{2M}\vec{\Phi}^{(+)}(z)' - \vec{\Phi}^{(-)}(z)^{T\,\prime}\frac{\hbar^2}{2M}\vec{\Phi}^{(+)}(z)\right]$$

$$= j_S(z) + j_B(z)$$

with

$$j_S(z) = \frac{1}{i\hbar}\left[\vec{\Phi}_S^{(-)}(z)^T \frac{\hbar^2}{2M_S}\vec{\Phi}_S^{(+)}(z)' - \vec{\Phi}_S^{(-)}(z)^{T\,\prime}\frac{\hbar^2}{2M_S}\vec{\Phi}_S^{(+)}(z)\right]$$

$$j_B(z) = \frac{1}{i\hbar}\left[\vec{\Phi}_B^{(-)}(z)^T \frac{\hbar^2}{2M_B}\vec{\Phi}_B^{(+)}(z)' - \vec{\Phi}_B^{(-)}(z)^{T\,\prime}\frac{\hbar^2}{2M_B}\vec{\Phi}_B^{(+)}(z)\right]$$

Assumption: There is some interaction between S and B only in the range $[z_<, z_>]$ with $z_2 > z_> \doteq z_< > z_1$.

By construction there holds $j(z) = j_S(z) + j_B(z) =$
$= \text{konst}$

In order to have $j_S(z_> + 0) = j_S(z_< - 0)$ there has to hold $j_B(z_> + 0) = j_B(z_< - 0)$!

Recall:
$$\frac{\hbar^2}{2M_B}\frac{d^2}{dz^2}\vec{\Phi}_B^{(+)}(z) = (V_{BB}(z) - E)\,\vec{\Phi}_B^{(+)}(z) + V_{BS}(z)\,\vec{\Phi}_S^{(+)}(z)$$
$$\frac{\hbar^2}{2M_B}\frac{d^2}{dz^2}\vec{\Phi}_B^{(-)}(z) = (V_{BB}^T(z) - E)\,\vec{\Phi}_B^{(-)}(z) + V_{SB}^T(z)\,\vec{\Phi}_S^{(-)}(z)$$

$$\Rightarrow \frac{d}{dz}\left[\frac{1}{i\hbar}\left(\vec{\Phi}_B^{(-)T}(z)\frac{\hbar^2}{2M_B}\vec{\Phi}_B^{(+)}(z)' - \vec{\Phi}_B^{(-)T}(z)'\frac{\hbar^2}{2M_B}\vec{\Phi}_B^{(+)}(z)\right)\right] =$$
$$= \left(\vec{\Phi}_B^{(-)T}(z)\,V_{BS}(z)\,\vec{\Phi}_S^{(+)}(z) - \vec{\Phi}_S^{(-)}(z)\,V_{SB}(z)\,\vec{\Phi}_B^{(+)}(z)\right)\frac{1}{i\hbar}$$

Recall: $$\frac{d}{dz}\, j_B(z) = \frac{1}{i\hbar}\Big(\vec{\Phi}_B^{(-)T}(z)\, V_{BS}(z)\, \vec{\Phi}_S^{(+)}(z) - \vec{\Phi}_S^{(-)T}(z)\, V_{SB}(z)\, \vec{\Phi}_B^{(+)}(z)\Big)$$

$$\Rightarrow 0 = j_B(z_>+0) - j_B(z_<-0)$$

$$= \int_{z_<-0}^{z_>+0} dz' \,\frac{d}{dz'} j_B(z') =$$

$$= \int_{z_<-0}^{z_>+0} dz' \,\frac{1}{i\hbar}\Big(\vec{\Phi}_B^{(-)T}(z')\, V_{BS}(z')\, \vec{\Phi}_S^{(+)}(z') - \vec{\Phi}_S^{(-)T}(z')\, V_{SB}(z')\, \vec{\Phi}_B^{(+)}(z')\Big)$$

$$= \int_{z_<-0}^{z_>+0} dz' \int_{z_1}^{z_2} dz'' \,\frac{1}{i\hbar} \sum_n \frac{1}{E-E_n}$$

$$\Big(\vec{\Phi}_S^{(+)T}(z')\, V_{BS}^T(z')\, \vec{\Phi}_{Bn}^{(-)}(z')\, \vec{\Phi}_{Bn}^{(+)T}(z'')\, V_{SB}^T(z'')\, \vec{\Phi}_S^{(-)}(z'')$$
$$- \vec{\Phi}_S^{(-)T}(z')\, V_{SB}(z')\, \vec{\Phi}_{Bn}^{(+)}(z')\, \vec{\Phi}_{Bn}^{(-)T}(z'')\, V_{BS}(z'')\, \vec{\Phi}_S^{(+)}(z'')\Big)$$

$$= \int_{z_<-0}^{z_>+0} dz' \int_{z_1}^{z_2} dz' \,\frac{1}{i\hbar} \sum_n \frac{1}{E-E_n}$$

$$\Big(\vec{\Phi}_S^{(-)}(z'')^T\, V_{SB}(z'')\, \vec{\Phi}_{Bn}^{(+)}(z'')\, \vec{\Phi}_{Bn}^{(-)}(z')\, V_{BS}(z')\, \vec{\Phi}_S^{(+)}(z')$$
$$- \vec{\Phi}_S^{(-)}(z')^T\, V_{SB}(z')\, \vec{\Phi}_{Bn}^{(+)}(z')\, \vec{\Phi}_{Bn}^{(-)}(z'')\, V_{BS}(z'')\, \Phi_S^{(+)}(z'')\Big)$$

$$\overset{!}{=} 0$$

?

Example: $V_{BS} = \delta(z-a_{BS})\, g_{BS}$

$$V_{SB} = \delta(z-a_{SB})\, g_{SB}$$

$$\Rightarrow \vec{\Phi}_B^{(-)}(a_{BS}) = \sum_n \int_{z_1}^{z_2} dz' \frac{\vec{\Phi}_{Bn}^{(-)}(a_{BS})\, \vec{\Phi}_{Bn}^{(+)}(z')^{T}\, \underbrace{}_{V_{SB}^T(z')}\, \vec{\Phi}_S^{(-)}(z')}{E-E_n}$$

$$= \sum_n \frac{\vec{\Phi}_{Bn}^{(-)}(a_{BS})\, \vec{\Phi}_{Bn}^{(+)}(a_{SB})^T g_{SB}^T\, \vec{\Phi}_S^{(-)}(a_{SB})}{E-E_n}$$

$$\vec{\Phi}_B^{(+)}(a_{SB}) = \sum_n \int_{z_1}^{z_2} dz' \frac{\vec{\Phi}_{Bn}^{(+)}(a_{SB})\, \vec{\Phi}_{Bn}^{(-)}(z')^T V_{BS}(z')\, \vec{\Phi}_S^{(+)}(z')}{E-E_n}$$

$$= \sum_n \frac{\vec{\Phi}_{Bn}^{(+)}(a_{SB})\, \vec{\Phi}_{Bn}^{(-)}(a_{BS})^T g_{BS}\, \vec{\Phi}_S^{(+)}(a_{BS})}{E-E_n}$$

$$j_B(z_>+0) - j_B(z_<-0) = \int_{z_<-0}^{z_>+0} dz' \Big(\vec{\Phi}_B^{(-)}(z')^T V_{BS}(z')\, \vec{\Phi}_S^{(+)}(z') - \vec{\Phi}_S^{(-)}(z')^T V_{SB}(z')\, \vec{\Phi}_B^{(+)}(z')\Big)\frac{1}{i\hbar}$$

$$= \Big(\vec{\Phi}_B^{(-)}(a_{BS})^T g_{BS}\, \vec{\Phi}_S^{(+)}(a_{BS}) - \vec{\Phi}_S^{(-)}(a_{SB})^T g_{SB}\, \vec{\Phi}_B^{(+)}(a_{SB})\Big]\frac{1}{i\hbar}$$

$$= \sum_n \frac{1}{E-E_n}\frac{1}{i\hbar}\cdot$$

$$\Big(\vec{\Phi}_S^{(-)}(a_{SB})^T g_{SB}\, \vec{\Phi}_{Bn}^{(+)}(a_{SB})\, \vec{\Phi}_{Bn}^{(-)}(a_{BS})^T g_{BS}\, \vec{\Phi}_S^{(+)}(a_{BS})$$

$$- \vec{\Phi}_S^{(-)}(a_{SB})^T g_{SB}\, \vec{\Phi}_{Bn}^{(+)}(a_{SB})\, \vec{\Phi}_{Bn}^{(-)}(a_{BS})^T g_{BS}\, \vec{\Phi}_S^{(+)}(a_{BS})\Big)$$

$$= 0$$

No problem with bound states!

? There should be discontinuity in ψ'

• $\Rightarrow$ Change ansatz for B-Greensfunction