Meson mass splittings in unquenched quark models

Tim Burns

Durham University, Dept of Mathematical Sciences

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Overview

1. Non-flip, triplet models
2. Meson mass splittings in unquenched quark models
3. Tests and applications

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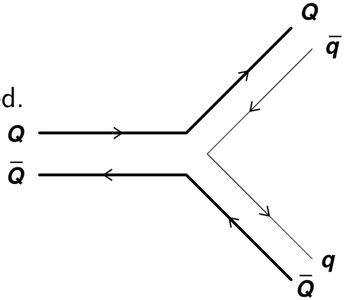
Non-flip, triplet models

Most models[1] involve the operator $\chi \cdot \mathbf{O}$, where

- χ creates a spin triplet $q\bar{q}$ pair
- $\mathbf{O}$ is the spatial part

and the spins of Q and $\overline{Q}$ are conserved.



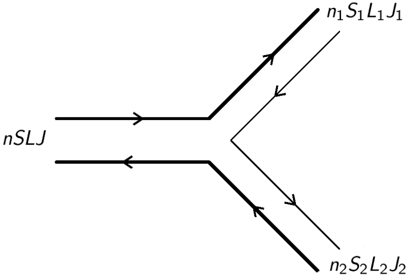
This includes:

- $^3P_0$ models
- flux tube models ($^3P_0$ and $^3S_1$)
- the Cornell model (Lorentz vector)
- microscopic models in the H.Q. limit (Lorentz scalar + vector)
- pseudoscalar-meson emission models

[1] TJB'14

Non-flip, triplet models: matrix element

We are interested in the matrix element $M_{jl}\begin{bmatrix} n & S & L & J \\ n_1 & S_1 & L_1 & J_1 \\ n_2 & S_2 & L_2 & J_2 \end{bmatrix}$



where

- n, S, L, J are radial, spin, orbital and total angular momenta,
- $j = J_1 \oplus J_2$,
- l is the partial wave.

Non-flip, triplet models: angular momentum coefficients

For non-flip, triplet models the matrix element factorises:

$$M_{jl}\begin{bmatrix} n & S & L & J \\ n_1 & S_1 & L_1 & J_1 \\ n_2 & S_2 & L_2 & J_2 \end{bmatrix} = \xi_{jl}\begin{bmatrix} S & L & J \\ S_1 & L_1 & J_1 \\ S_2 & L_2 & J_2 \end{bmatrix} \cdot \mathbf{A}_l\begin{bmatrix} n & L \\ n_1 & L_1 \\ n_2 & L_2 \end{bmatrix}$$

- **A** contains the spatial part **O** (model-dependent), and
- ξ contains the spin part χ (model-independent).

Orthogonality of the ξ coefficients gives a "closure" relation

$$\sum_{\substack{S_1 J_1 \\ S_2 J_2 \\ j}} M_{jl}\begin{bmatrix} \widehat{n} & \widehat{S} & \widehat{L} & J \\ n_1 & S_1 & L_1 & J_1 \\ n_2 & S_2 & L_2 & J_2 \end{bmatrix}^* M_{jl}\begin{bmatrix} n & S & L & J \\ n_1 & S_1 & L_1 & J_1 \\ n_2 & S_2 & L_2 & J_2 \end{bmatrix} = \delta_{\widehat{S}S}\delta_{\widehat{L}L}\mathbf{A}_l^*\begin{bmatrix} \widehat{n} & \widehat{L} \\ n_1 & L_1 \\ n_2 & L_2 \end{bmatrix} \cdot \mathbf{A}_l\begin{bmatrix} n & L \\ n_1 & L_1 \\ n_2 & L_2 \end{bmatrix}$$

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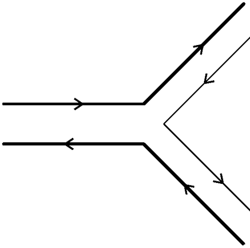
Mesons

The coupling



implies that physical states $\left|i\right\rangle$ are admixtures of

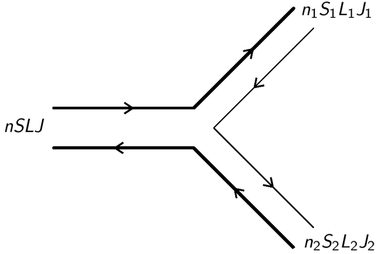
- bare (valence) states $Q\overline{Q}$, and
- meson-meson continua $(Q\overline{q})(q\overline{Q})$

In solving for the eigenvalues E_i the key quantity is the matrix element:

$$\left\langle \widehat{n}\widehat{S}\widehat{L}J\middle|\Omega(E_i)\middle|nSLJ\right\rangle = \sum_{\substack{n_1S_1L_1J_1\\ n_2S_2L_2J_2\\ jl}} \int dpp^2 \frac{M_{jl}\begin{bmatrix}\widehat{n} & \widehat{S} & \widehat{L} & J\\ n_1 & S_1 & L_1 & J_1\\ n_2 & S_2 & L_2 & J_2\end{bmatrix}^* M_{jl}\begin{bmatrix}n & S & L & J\\ n_1 & S_1 & L_1 & J_1\\ n_2 & S_2 & L_2 & J_2\end{bmatrix}}{E_{12}(p) - E_i}$$

Mesons

The coupling



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Mesons: degenerate continua

With no spin splittings among the continua,

$$\left\langle \hat{n}\hat{S}\hat{L}J \middle| \Omega(E_i) \middle| nSLJ \right\rangle = \delta_{\hat{S}S}\delta_{\hat{L}L}\left\langle \hat{n}L \middle| \Omega(E_i) \middle| nL \right\rangle$$

$$\left\langle \hat{n}L \middle| \Omega(E_i) \middle| nL \right\rangle = \sum_{\substack{n_1 L_1 \\ n_2 L_2 \\ l}} \int dp p^2 \frac{\mathbf{A}_l^* \begin{bmatrix} \hat{n} & L \\ n_1 & L_1 \\ n_2 & L_2 \end{bmatrix} \cdot \mathbf{A}_l \begin{bmatrix} n & L \\ n_1 & L_1 \\ n_2 & L_2 \end{bmatrix}}{E_{12}(p) - E_i}$$

- S and L are good quantum numbers (cf. pert theory [2])

With no spin splittings among the continuum *or* valence states

- physical masses are independent of S and J (cf. pert theory)
- configuration mixing is independent of S and J.

[2] Barnes and Swanson '08

Mesons: degenerate continua, split valence states

Allowing for spin splittings among the valence masses,

$$M_{nSLJ} = M_{nL} + \delta M_{nSLJ}$$

we anticipate splittings among the physical masses,

$$E_{nSLJ} = E_{nL} + \delta E_{nSLJ}$$

Ignoring mixing, the mass shift formula (below threshold) is

$$E_{nSLJ} = M_{nSLJ} - \left\langle \Omega(E_{nSLJ}) \right\rangle_{nSLJ}$$

and the valence component is

$$Z_{nSLJ} = \frac{1}{1 + \left\langle \omega(E_{nSLJ}) \right\rangle_{nSLJ}}$$

$$\omega(E_{nSLJ}) = \frac{\partial \Omega(E_{nSLJ})}{\partial E_{nSLJ}}$$

Taylor expanding,

$$\Omega(E_{nSLJ}) \approx \Omega(E_{nL}) + \delta E_{nSLJ} \frac{\partial \Omega(E_{nL})}{\partial E_{nL}} = \Omega(E_{nL}) + \delta E_{nSLJ}\, \omega(E_{nL})$$

Mesons: degenerate continua, split valence states

With no spin splittings among the continua,

$$\big\langle \Omega(E_{nSLJ}) \big\rangle_{nSLJ} \approx \big\langle \Omega(E_{nL}) \big\rangle_{nL} + \delta E_{nSLJ} \big\langle \omega(E_{nL}) \big\rangle_{nL}$$

Separating out the spin-averaged mass shift

$$E_{nL} = M_{nL} - \big\langle \Omega(E_{nL}) \big\rangle_{nL}$$

gives a relation between the physical and valence spin splittings

$$\delta E_{nSLJ} = Z_{nL} \delta M_{nSLJ}$$

where the spin-averaged valence component is

$$Z_{nL} = \frac{1}{1 + \big\langle \omega(E_{nL}) \big\rangle_{nL}}$$

Mesons: degenerate continua, split valence states

With no spin splittings among the continua,

$$\big\langle \Omega(E_{nSLJ}) \big\rangle_{nSLJ} \approx \big\langle \Omega(E_{nL}) \big\rangle_{nL} + \delta E_{nSLJ} \big\langle \omega(E_{nL}) \big\rangle_{nL}$$

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Mesons: split continua, split valence states

What if we incorporate spin splittings among the continua?
For S-wave continua ($L_1 = L_2 = 0$) with

- centre of mass m
- hyperfine splitting δ

the physical splittings are computed in a power series in δ/m, using properties of ξ coefficients.

For S-wave mesons ($L = 0$), terms of order $\mathcal{O}(\delta/m)$ make no contribution to the spin splitting!

For orbitally excited mesons ($L \geqslant 1$), terms of order $\mathcal{O}(\delta/m)$ are

- small due to cancellations, and
- proportional to $\langle \mathbf{L} \cdot \mathbf{S} \rangle_{SLJ}$.

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Hyperfine splitting ($L = 0$): models

Below are examples of the model[3] calculations showing

- spin-averaged mass shifts $\langle \Omega \rangle_{nS}$
- bare hyperfine splitting δM
- physical hyperfine splitting δE
- predicted hyperfine splitting $\delta E^{pred.}$

	$\langle \Omega \rangle_{nS}$	δM	δE	$\delta E^{pred.}$
$c\bar{c}$				
1S (K)	174	129	117	
2S (K)	212	64	48	
$b\bar{b}$				
1S (LD)	57.41	71.39	68.50*	
2S (LD)	67.58	23.12	21.30	
3S (LD)	67.74	15.73	14.00	

[3] K=Kalashnikova, LD=Liu &Ding

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- physical hyperfine splitting δE
- predicted hyperfine splitting $\delta E^{pred.}$

	$\langle \Omega \rangle_{nS}$	δM	δE	$\delta E^{pred.}$
$c\bar{c}$				
1S (K)	174	129	117	116.4
2S (K)	212	64	48	48.4
$b\bar{b}$				
1S (LD)	57.41	71.39	68.50*	68.44
2S (LD)	67.58	23.12	21.30	21.36
3S (LD)	67.74	15.73	14.00	14.06

[3] K=Kalashnikova, LD=Liu &Ding

Hyperfine splitting ($L = 0$): relation to e^+e^- width

In the quenched quark model, the hyperfine splitting and e^+e^- widths are both $\propto |R_{nS}(0)|^2$, which leads to the model-independent relation

$$\frac{\delta M_{2S}}{\delta M_{1S}} = \frac{\Gamma_{e^+e^- \to 2^3S_1(bare)}}{\Gamma_{e^+e^- \to 1^3S_1(bare)}}$$

How is this modified by unquenching? [4] The physical mass splitting is scaled downwards,

$$\delta E_{nS} = Z_{nS}\delta M_{nS}$$

but so is the e^+e^- width

$$\Gamma_{e^+e^- \to n^3S_1(phys.)} \to Z_{n^3S_1}\Gamma_{e^+e^- \to n^3S_1(bare)}$$

To a very good approximation $Z_{n^3S_1} \approx Z_{nS}$, so the relation survives:

$$\frac{\delta E_{2S}}{\delta E_{1S}} = \frac{\Gamma_{e^+e^- \to 2^3S_1(phys.)}}{\Gamma_{e^+e^- \to 1^3S_1(phys.)}}$$

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[4] TJB '12

Hyperfine splitting ($L = 0$): relation to e^+e^- width

The relation is well-satisfied for charmonia

$$\frac{\delta E_{2S}}{\delta E_{1S}} = 0.407 \pm 0.015$$
$$\frac{\Gamma_{e^+e^- \to \psi'}}{\Gamma_{e^+e^- \to J/\psi}} = 0.423 \pm 0.018$$

and, with the (now established) Belle $\eta_b(2S)$, bottomonia

$$\frac{\delta E_{2S}}{\delta E_{1S}} = 0.42 \pm 0.09$$
$$\frac{\Gamma_{e^+e^- \to \Upsilon'}}{\Gamma_{e^+e^- \to \Upsilon}} = 0.457 \pm 0.014$$

Predictions for $\eta_b(3S)$:

$$\delta M_{3S} = 20.6 \pm 1.7 \ MeV \qquad M_{\eta_b(3S)} = 10334.6 \pm 2.2 \ MeV$$

Hyperfine splitting ($L \neq 0$): experiment

The quenched quark model result,

$$\frac{1}{9}\left(M_{{}^3P_0} + 3M_{{}^3P_1} + 5M_{{}^3P_2}\right) - M_{{}^1P_1} = 0$$

is well-satisfied by charmonia and bottomonia:

$$\overline{M}_{\chi_c(1P)} - M_{h_c(1P)} = -0.05 \pm 0.19 \pm 0.16\ MeV$$
$$\overline{M}_{\chi_b(1P)} - M_{h_b(1P)} = +2 \pm 4 \pm 1\ MeV \qquad (BaBar)$$
$$\overline{M}_{\chi_b(1P)} - M_{h_b(1P)} = +1.62 \pm 1.52\ MeV \qquad (Belle)$$
$$\overline{M}_{\chi_b(2P)} - M_{h_b(2P)} = +0.48^{+1.57}_{-1.22}\ MeV \qquad (Belle)$$

A priori one might expect large corrections due to large, different mass shifts, but the relation $\delta E_{nSLJ} = Z_{nL}\delta M_{nSLJ}$ protects the result[5].

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[5] TJB '11

Hyperfine splitting ($L \neq 0$): models[6]

	$\langle\Omega\rangle_{^3P_0}$	$\langle\Omega\rangle_{^3P_1}$	$\langle\Omega\rangle_{^3P_2}$	$\langle\Omega\rangle_{^1P_1}$	*Induced*
$c\bar{c}$					
1P (BS)	459	496	521	504	
1P (K)	198	215	228	219	
1P (LMC)	35	38	63	52	
1P (YLCD)	131	152	175	162	
1P (OT)	173	180	185	182	
$b\bar{b}$					
1P (OT)	43	44	45	44	
2P (OT)	55	56	58	57	
1P (FS)	108	114	117	115	
2P (FS)	137	144	149	146	
1P (LD)	80.777	84.823	87.388	85.785	
2P (LD)	73.578	77.608	80.146	78.522	

[6] BS=Barnes & Swanson, K=Kalashnikova, LMC=Li, Meg & Chao, YLCD=Yang, Li, Chen & Deng, OT=Ono& Tornqvist, LD=Liu &Ding, FS=Ferretti & Santopinto

Hyperfine splitting ($L \neq 0$): models[6]

	$\langle\Omega\rangle_{^3P_0}$	$\langle\Omega\rangle_{^3P_1}$	$\langle\Omega\rangle_{^3P_2}$	$\langle\Omega\rangle_{^1P_1}$	*Induced*
$c\bar{c}$					
1P (BS)	459	496	521	504	− 1.8
1P (K)	198	215	228	219	− 1.3
1P (LMC)	35	38	63	52	− 2.9
1P (YLCD)	131	152	175	162	− 0.4
1P (OT)	173	180	185	182	− 0.0
$b\bar{b}$					
1P (OT)	43	44	45	44	− 0.4
2P (OT)	55	56	58	57	− 0.0
1P (FS)	108	114	117	115	− 0.0
2P (FS)	137	144	149	146	− 0.0
1P (LD)	80.777	84.823	87.388	85.785	− 0.013
2P (LD)	73.578	77.608	80.146	78.522	− 0.048

[6] BS=Barnes & Swanson, K=Kalashnikova, LMC=Li, Meg & Chao, YLCD=Yang, Li, Chen & Deng, OT=Ono& Tornqvist, LD=Liu &Ding, FS=Ferretti & Santopinto

Spin-orbit and tensor splittings ($L \neq 0$): models

Spin-orbit (LS) and tensor (T) splittings are also scaled by Z_{nL}.

	$\langle \Omega \rangle$	δM_{LS}	δE_{LS}	$\delta E_{LS}^{pred.}$	δM_T	δE_T	$\delta E_T^{pred.}$
$c\bar{c}$							
1P (K)	220	43	34.8		−23.3	−20.4	
1P (OT)	182	35.8	32.5		−22.1	−20.8	
$b\bar{b}$							
1P (LD)	85.84	16.31	14.75		−6.05	−5.58	
2P (LD)	78.92	12.99	11.46		−4.78	−4.35	
3P (LD)	80.15	11.75	9.86		−4.23	−3.85	
1D (LD)	97.96	2.44	2.07		−1.26	−1.17	
2D (LD)	89.36	2.36	2.03		−1.17	−1.16	

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$c\bar{c}$							
1P (K)	220	43	34.8	35.1	−23.3	−20.4	− 19.0
1P (OT)	182	35.8	32.5	33.2	−22.1	−20.8	− 20.5
$b\bar{b}$							
1P (LD)	85.84	16.31	14.75	14.84	−6.05	−5.58	− 5.50
2P (LD)	78.92	12.99	11.46	11.49	−4.78	−4.35	− 4.23
3P (LD)	80.15	11.75	9.86	9.76	−4.23	−3.85	− 3.51
1D (LD)	97.96	2.44	2.07	2.12	−1.26	−1.17	− 1.10
2D (LD)	89.36	2.36	2.03	1.97	−1.17	−1.16	− 0.98

Spin-orbit and tensor splittings ($L \neq 0$): the ratio R

The ratio

$$R = \frac{M_{^3P_2} - M_{^3P_1}}{M_{^3P_1} - M_{^3P_0}}$$

- is sensitive to LS and T, and
- can be used to infer the properties of the $Q\overline{Q}$ potential

In the unquenching formula, R is invariant, but in practice it generally decreases due to unquenching.

$c\overline{c}$	$R^{(bare)}$	$R^{(phys.)}$	$b\overline{b}$	$R^{(bare)}$	$R^{(phys.)}$
1P (LMC)	0.65	0.57	LD (1P)	0.736	0.723
1P (K)	0.51	0.47	LD (2P)	0.741	0.722
1P (FS)	0.59	0.51	LD (3P)	0.753	0.705
1P (OT)	0.44	0.42	OT (1P)	0.913	0.909
1P (YLCD)	0.59	0.47	OT (2P)	0.894	0.833

Splittings in lattice QCD

In unquenched lattice QCD, spin splittings vary with dynamical quark masses. We might expect

$$m_q \downarrow \Longrightarrow \downarrow \epsilon \ (and \ \uparrow M_{jl}) \implies Z_{nL} \downarrow \Longrightarrow E_{nSLJ} \downarrow$$

but it is not so simple:

- changing m_q influences α_s, and
- unquenched lattice QCD with $Q\overline{Q}$ operators but not $(Q\overline{q})(q\overline{Q})$ operators are not sensitive to $Q\overline{Q} \to (Q\overline{q})(q\overline{Q})$[7].

Lattice QCD with (complete multiplets of) $(Q\overline{q})(q\overline{Q})$ operators could test $E_{nSLJ} = Z_{nL}\delta M_{nSLJ}$, measuring splittings and Z-factors at various quark masses. Bali *et al.*[8] have a single data point for each of several $c\overline{c}$, with a limited set of continua.

[7] See talks Lang, Rupp; Dudek, Edwards & Thomas '12

[8] Bali, Collins & Ehmann '11

Valence component inequalities

Expanding $\omega(E_{nSLJ})$ in a similar way to $\Omega(E_{nSLJ})$, leads to inequalities among the Z-factors, e.g.

$$Z_{^1S_0} > Z_{^3S_1}$$

$$Z_{^3P_0} > Z_{^3P_1} > Z_{^3P_2}$$

Relations among leptonic widths, arising from cancelling $\left|R_{nL}(0)\right|^2$ factors, are modified, e.g.

$$\frac{\Gamma_{^1S_0 \to \gamma\gamma}}{\Gamma_{^3S_1 \to e^+e^-}} > \frac{4}{3}\left(1 + 1.96\frac{\alpha_s}{\pi}\right)$$

$$\frac{\Gamma_{^3P_2 \to \gamma\gamma}}{\Gamma_{^3P_0 \to \gamma\gamma}} < \frac{4}{15}\left(1 - 5.51\frac{\alpha_s}{\pi}\right)$$

Final remarks: everything follows from $\chi \cdot \mathbf{O}$

With no spin splitting, previous results in pert. theory

- generalise to full coupled-channel problem,
- generalise to all models,
- apply automatically to hybrids,
- extend to deal with mixing.

Including spin splittings, the formula $\delta E_{nSLJ} = Z_{nL} \delta M_{nSLJ}$

- agrees with model calculations (and common observations),
- protects e^+e^- width formulae for $L = 0$,
- protects zero hyperfine for $L \neq 0$,
- leaves R approximately invariant,
- might be testable on lattice QCD,

. . . and is useful for calculations.

Extra slides

Angular momentum coefficients: some applications

The ξs are model-independent and can be used for

- strong decays (amplitudes and widths)
- unquenched quark models (mass shifts, mixing angles, continuum wavefunctions)

As well as ordinary transitions, they apply to

- hybrids in the flux tube and constituent gluon model, and
- spin-mixed $({}^1L_L - {}^3L_L)$ states

Their symmetries imply

- the conservation of C and G parity, and
- new selection rules (e.g. spin triplet selection rule).

Angular momentum coefficients: some applications

The ξs also lead to predictions for

- ratios of amplitudes, e.g. for ${}^3S_1 \to {}^3S_1{}^3S_1$,

$$M_{{}^5P}/M_{{}^1P} = -2\sqrt{5}$$

- ratios-of-ratios, e.g.[9]

$$\frac{M_D}{M_S}(a_1 \to \rho\pi) \Big/ \frac{M_D}{M_S}(b_1 \to \omega\pi) = -\frac{1}{2}$$

- relations among widths, e.g. for decays to ${}^3S_1{}^1S_0$

$$35\Gamma[{}^3F_3] = 50\Gamma[{}^3F_2] + 27\Gamma[{}^3F_4]$$

- relations testable on the lattice, e.g. for the exotic π_1 hybrid[10]

$$M_S(\pi_1 \to b_1\pi)/M_S(\pi_1 \to f_1\pi) = 2$$

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[9] TJB, Close, Thomas '07

[10] TJB, Close '06

Hyperfine splitting ($L \neq 0$): models

The same is true for D- and F-wave hyperfine splittings.

	$\langle\Omega\rangle_{^3L_{L-1}}$	$\langle\Omega\rangle_{^3L_L}$	$\langle\Omega\rangle_{^3L_{L+1}}$	$\langle\Omega\rangle_{^1L_L}$	*Induced*
$b\bar{b}$					
1D (FS)	159	161	163	161	
1D (LD)	96.793	97.692	98.696	97.890	
2D (LD)	88.412	89.078	90.041	89.239	
1F (LD)	102.945	103.043	103.398	103.048	

From the BaBar data on $^3D_1$, $^3D_2$ and $^3D_3$ bottomonia, we can predict the $^1D_2$ mass[11]

$$M_{\eta_{b2(1D)}} = 10165.84 \pm 1.8 \ MeV$$

[11] TJB *et al.* '10

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$b\bar{b}$					
1D (FS)	159	161	163	161	-0.5
1D (LD)	96.793	97.692	98.696	97.890	-0.091
2D (LD)	88.412	89.078	90.041	89.239	-0.155
1F (LD)	102.945	103.043	103.398	103.048	-0.123

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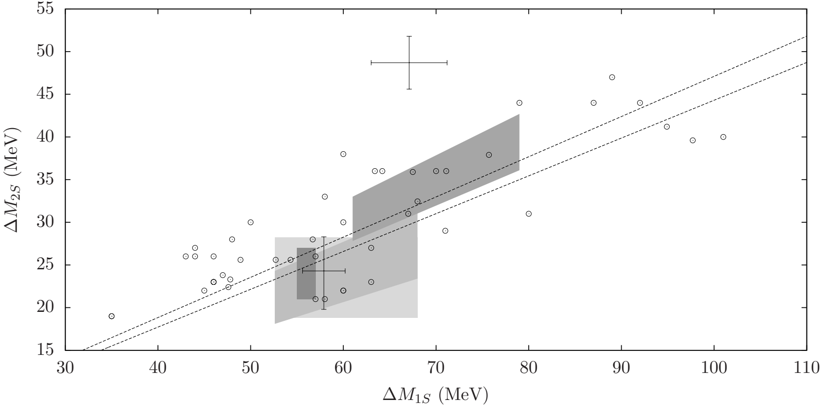
Hyperfine splitting ($L = 0$): relation to e^+e^- width

Candidates for $\eta_b(2S)$ vs. theory predictions[12]:



[12] TJB '12